CREATION AND ANNIHILATION OPERATORS FOR THE 3-D HARMONIC OSCILLATOR

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Reference: Tom Lancaster and Stephen J. Blundell, *Quantum Field Theory for the Gifted Amateur*, (Oxford University Press, 2014), Problem 3.3.

For the 3-d harmonic oscillator, we can write the hamiltonian as

$$\hat{H}=\frac{1}{2m}\left(\hat{p}_1^2+\hat{p}_2^2+\hat{p}_3^2\right)+\frac{m\omega^2}{2}\left(\hat{x}_1^2+\hat{x}_2^2+\hat{x}_3^2\right) \tag{1}$$

This is effectively three independent oscillators, one in each of the three coordinate directions, so using the 1-d form of the hamiltonian in terms of creation and annihilation operators, we have

$$\hat{H}=\hbar\omega\sum_{i=1}^{3}\left(\frac{1}{2}+\hat{a}_i^{\dagger}\hat{a}_i\right) \tag{2}$$

Using the position and momentum operators expressed as

$$\hat{x}_i = \sqrt{\frac{\hbar}{2m\omega}}\left(\hat{a}_i^{\dagger}+\hat{a}_i\right) \tag{3}$$

$$\hat{p}_i = i\sqrt{\frac{\hbar m\omega}{2}}\left(\hat{a}_i^{\dagger}-\hat{a}_i\right) \tag{4}$$

we can write the components of angular momentum in terms of creation and annihilation operators. For example, the z component can be written as

$$L^3=\hat{x}_1\hat{p}_2-\hat{x}_2\hat{p}_1 \tag{5}$$

$$=\frac{i\hbar}{2}\left[\left(\hat{a}_1^{\dagger}+\hat{a}_1\right)\left(\hat{a}_2^{\dagger}-\hat{a}_2\right)-\left(\hat{a}_2^{\dagger}+\hat{a}_2\right)\left(\hat{a}_1^{\dagger}-\hat{a}_1\right)\right] \tag{6}$$

$$=\frac{i\hbar}{2}\left[-\hat{a}_1^{\dagger}\hat{a}_2+\hat{a}_1\hat{a}_2^{\dagger}-\hat{a}_2\hat{a}_1^{\dagger}+\hat{a}_2^{\dagger}\hat{a}_1\right] \tag{7}$$

$$=-i\hbar\left(\hat{a}_1^{\dagger}\hat{a}_2-\hat{a}_2^{\dagger}\hat{a}_1\right) \tag{8}$$

where we've used the fact that all operators of one coordinate commute with all operators of the other two coordinates. If we work out the other two coordinates we get the general formula

$$L^i = -i\hbar\varepsilon^{ijk}\hat{a}_j^\dagger \hat{a}_k \tag{9}$$

where ε^{ijk} is the Levi-Civita symbol, which is +1 if ijk is a even permutation of 1,2,3, -1 if ijk is an odd permutation of 1,2,3 and 0 if any two of ijk are equal, and repeated indices are summed.

We can define some new creation and annihilation operators as follows:

$$\begin{aligned}
\hat{b}_1^\dagger &\equiv -\frac{1}{\sqrt{2}}\left(\hat{a}_1^\dagger + i\hat{a}_2^\dagger\right) && (10)\\
\hat{b}_0^\dagger &\equiv \hat{a}_3^\dagger && (11)\\
\hat{b}_{-1}^\dagger &\equiv \frac{1}{\sqrt{2}}\left(\hat{a}_1^\dagger - i\hat{a}_2^\dagger\right) && (12)
\end{aligned}$$

$$\begin{aligned}
\hat{b}_1 &\equiv -\frac{1}{\sqrt{2}}\left(\hat{a}_1 - i\hat{a}_2\right) && (13)\\
\hat{b}_0 &\equiv \hat{a}_3 && (14)\\
\hat{b}_{-1} &\equiv \frac{1}{\sqrt{2}}\left(\hat{a}_1 + i\hat{a}_2\right) && (15)
\end{aligned}$$

From the commutation relations for $\hat{a}_i^\dagger$ and $\hat{a}_i$:

$$\left[\hat{a}_i, \hat{a}_j^\dagger\right] = \delta_{ij} \tag{16}$$

we get

$$\left[\hat{b}_0, \hat{b}_j^\dagger\right] = \delta_{0,j} \tag{17}$$

$$\left[\hat{b}_1, \hat{b}_1^\dagger\right] = \frac{1}{2}\left(\left[\hat{a}_1, \hat{a}_1^\dagger\right] + (-i)\, i \left[\hat{a}_2, \hat{a}_2^\dagger\right]\right) \tag{18}$$

$$= \frac{1}{2}(1+1) \tag{19}$$

$$= 1 \tag{20}$$

$$\left[\hat{b}_1, \hat{b}_{-1}^\dagger\right] = -\frac{1}{2}\left(\left[\hat{a}_1, \hat{a}_1^\dagger\right] + i^2 \left[\hat{a}_2, \hat{a}_2^\dagger\right]\right) \tag{21}$$

$$= 0 \tag{22}$$

$$\left[\hat{b}_{-1}, \hat{b}_{-1}^\dagger\right] = \frac{1}{2}\left(\left[\hat{a}_1, \hat{a}_1^\dagger\right] + (-i)\, i \left[\hat{a}_2, \hat{a}_2^\dagger\right]\right) \tag{23}$$

$$= 1 \tag{24}$$

$$\left[\hat{b}_{-1}, \hat{b}_1^\dagger\right] = -\frac{1}{2}\left(\left[\hat{a}_1, \hat{a}_1^\dagger\right] + i^2 \left[\hat{a}_2, \hat{a}_2^\dagger\right]\right) \tag{25}$$

$$= 0 \tag{26}$$

so in general

$$\left[\hat{b}_i, \hat{b}_j^\dagger\right] = \delta_{ij} \tag{27}$$

To express the hamiltonian 2 in terms of the new operators, we need

$$\hat{b}_1^\dagger \hat{b}_1 = \frac{1}{2}\left[\hat{a}_1^\dagger \hat{a}_1 + \hat{a}_2^\dagger \hat{a}_2 + i\left(\hat{a}_2^\dagger \hat{a}_1 - \hat{a}_1^\dagger \hat{a}_2\right)\right] \tag{28}$$

$$\hat{b}_{-1}^\dagger \hat{b}_{-1} = \frac{1}{2}\left[\hat{a}_1^\dagger \hat{a}_1 + \hat{a}_2^\dagger \hat{a}_2 - i\left(\hat{a}_2^\dagger \hat{a}_1 - \hat{a}_1^\dagger \hat{a}_2\right)\right] \tag{29}$$

$$\hat{b}_1^\dagger \hat{b}_1 + \hat{b}_{-1}^\dagger \hat{b}_{-1} = \hat{a}_1^\dagger \hat{a}_1 + \hat{a}_2^\dagger \hat{a}_2 \tag{30}$$

$$\hat{H} = \hbar\omega \sum_{i=-1}^{+1}\left(\frac{1}{2} + \hat{b}_i^\dagger \hat{b}_i\right) \tag{31}$$

Finally, we can write $\hat{L}^3$ in terms of the new operators.

$$-\hat{b}_{-1}^\dagger \hat{b}_{-1} + 0 \times \hat{b}_0^\dagger \hat{b}_0 + \hat{b}_1^\dagger \hat{b}_1 = i\left(\hat{a}_2^\dagger \hat{a}_1 - \hat{a}_1^\dagger \hat{a}_2\right) \tag{32}$$

$$\hat{L}^3 = \hbar \sum_{m=-1}^{+1} m \hat{b}_m^\dagger \hat{b}_m \tag{33}$$