FERMION WAVE FUNCTIONS; THE SLATER DETERMINANT

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Reference: Tom Lancaster and Stephen J. Blundell, *Quantum Field Theory for the Gifted Amateur*, (Oxford University Press, 2014), Problem 3.4.

We've seen that a system consisting of three fermions has a wave function

$$\begin{aligned}\sqrt{6}\psi_f(x_a,x_b,x_c) = {} & \psi_{1a}\psi_{2b}\psi_{3c}+\psi_{1b}\psi_{2c}\psi_{3a}+\psi_{1c}\psi_{2a}\psi_{3b}- \\ & \psi_{1a}\psi_{2c}\psi_{3b}-\psi_{1c}\psi_{2b}\psi_{3a}-\psi_{1b}\psi_{2a}\psi_{3c}\end{aligned} \tag{1}$$

where ψ_{1a} is the wave function for a particle at location x_a in state ψ_1 and so on. The $\sqrt{6}$ is for normalization. If we find the inner product of ψ_f with itself, then the inner product of each of the 6 terms on the RHS with itself contributes a 1, while the inner product of a term on the RHS with a different term on the RHS is always zero due to the orthogonality of the different wave functions, and the fact that inner products are taken over functions that use the same coordinates. For example, the inner product of the first term on the RHS with the second term is

$$\langle\psi_{1a}\psi_{2b}\psi_{3c}|\psi_{1b}\psi_{2c}\psi_{3a}\rangle = \langle\psi_{1a}|\psi_{3a}\rangle\langle\psi_{2b}|\psi_{1b}\rangle\langle\psi_{3c}|\psi_{2c}\rangle \tag{2}$$

and each of the inner products on the RHS here is zero because ψ_1 is orthogonal to ψ_3 and so on.

To get the general wave function for N fermions, we need a wave function that is antisymmetric under the exchange of any two coordinates. One property of a determinant of a matrix is that it is antisymmetric under the exchange of any two rows or any two columns. Also, each term in the expansion of a determinant contains one factor from each row and each column. If we have N particles and write the wave function as a Slater determinant, we have

$$\psi_f(x_{r_1}x_{r_2}\ldots x_{r_N}) = \frac{1}{\sqrt{N!}}\begin{vmatrix}\psi_{1r_1} & \psi_{2r_1} & \cdots & \psi_{Nr_1}\\ \psi_{1r_2} & \psi_{2r_2} & \cdots & \psi_{Nr_2}\\ \vdots & \vdots & \ddots & \vdots\\ \psi_{1r_N} & \psi_{2r_N} & \cdots & \psi_{Nr_N}\end{vmatrix} \tag{3}$$

The Slater determinant has $N!$ terms in its expansion. To see this, expand about the first row, where there are N elements. Each element is multiplied by the corresponding sub-determinant which is of size $(N-1)\times(N-1)$ and so on so you get $N\times(N-1)\times\ldots 1=N!$ terms. As with the three particle case above, each term contributes 1 to the inner product $\left\langle \psi_f \middle| \psi_f \right\rangle$, so we need to divide the determinant by $\sqrt{N!}$ to normalize it.