POISSON BRACKETS, COMMUTATORS AND JACOBI IDENTITY

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog.

Reference: Tom Lancaster and Stephen J. Blundell, *Quantum Field Theory for the Gifted Amateur*, (Oxford University Press, 2014), Problem 5.2.

Post date: 5 Mar 2019.

In classical mechanics, the Poisson bracket is defined as

$$\{A,B\} \equiv \frac{\partial A}{\partial q_i}\frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i}\frac{\partial B}{\partial q_i} \tag{1}$$

where a sum over i is assumed, and q_i is a generalized coordinate with p_i its canonical momentum, defined from the Lagrangian as

$$p_i = \frac{\partial L}{\partial \dot{q}_i} \tag{2}$$

From 1, if we swap A and B, we have

$$\{B,A\} = \frac{\partial B}{\partial q_i}\frac{\partial A}{\partial p_i} - \frac{\partial B}{\partial p_i}\frac{\partial A}{\partial q_i} \tag{3}$$

$$= -\left(\frac{\partial A}{\partial q_i}\frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i}\frac{\partial B}{\partial q_i}\right) \tag{4}$$

$$= -\{A,B\} \tag{5}$$

We can also derive the Jacobi identity as follows. The identity to be proved is:

$$\{\{A,B\},C\} + \{\{C,A\},B\} + \{\{B,C\},A\} = 0 \tag{6}$$

To simplify the notation, I'll define

$$A_q \equiv \frac{\partial A}{\partial q_i} \tag{7}$$

$$A_p \equiv \frac{\partial A}{\partial p_i} \tag{8}$$

with similar notations for second derivatives, such as $A_{pq} \equiv \frac{\partial^2 A}{\partial p \partial q}$ and so on.

We start with

$$\{\{A,B\},C\} = \{A_q B_p - A_p B_q, C\} \tag{9}$$

We can now use the identity

$$\{\lambda\sigma, C\} = \lambda\{\sigma, C\} + \sigma\{\lambda, C\} \tag{10}$$

We get

$$\{\{A,B\},C\} = A_q\{B_p,C\} + B_p\{A_q,C\} - A_p\{B_q,C\} - B_q\{\partial_p A, C\} \tag{11}$$

By cyclic permutation, we have also

$$\{\{C,A\},B\} = C_q\{A_p,B\} + A_p\{C_q,B\} - C_p\{A_q,B\} - A_q\{C_p,B\} \tag{12}$$

$$\{\{B,C\},A\} = B_q\{C_p,A\} + C_p\{B_q,A\} - B_p\{C_q,A\} - C_q\{B_p,A\} \tag{13}$$

We can now take the sum of these 3 equations, and do so in groups of 4 terms on the RHS. Consider first the sum of the first and third terms from 11 with the second and fourth terms from 13. This gives

Remember that the order of second derivatives doesn't matter, so that $B_{pq} = B_{qp}$.

$$\begin{aligned} A_q\{B_p,C\} - A_p\{B_q,C\} + C_p\{B_q,A\} - C_q\{B_p,A\} = \\ A_q(B_{pq}C_p - B_{pp}C_q) - A_p(B_{qq}C_p - B_{qp}C_q) + \\ C_p(B_{qq}A_p - B_{pq}A_q) - C_q(B_{pq}A_p - B_{pp}A_q) \end{aligned} \tag{14}$$

Looking at the RHS, we can see that all the terms cancel in pairs, so this first sum of four terms is zero. In a similar way, we can (if you really want to) expand the other terms to find that the sum of the second and fourth terms from 11 together with the first and third terms from 12 again cancel in pairs to give zero, and finally the sum of the second and fourth terms from 12 together with the first and third terms from 13 also give zero. Thus we've verified the Jacobi identity 6.

Similar relations also hold for quantum mechanical commutators. It's fairly obvious that

$$[A,B] = AB - BA = -[B,A] \tag{15}$$

The Jacobi identity for commutators is fortunately easier to prove since it doesn't involve derivatives. We have

$$[[A,B],C] = ABC - BAC - CAB + CBA \tag{16}$$
$$[[C,A],B] = CAB - ACB - BCA + BAC \tag{17}$$
$$[[B,C],A] = BCA - CBA - ABC + ACB \tag{18}$$

Adding up all three rows, we again see that all the terms cancel in pairs so that

$$[[A,B],C] + [[C,A],B] + [[B,C],A] = 0 \tag{19}$$