MAXIMUM SPACE-TIME PATH FOR FREE PARTICLE

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Reference: Tom Lancaster and Stephen J. Blundell, *Quantum Field Theory for the Gifted Amateur*, (Oxford University Press, 2014), Problem 5.5.

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The Lagrangian for a relativistic free particle is

$$L=-\frac{mc^{2}}{\gamma} \tag{1}$$

The action integral is therefore

$$S=\int_{\tau_{1}}^{\tau_{2}}L\gamma\,d\tau \tag{2}$$

where τ is the particle's proper time, so that the time is $t=\gamma\tau$. The invariant interval ds between two spacetime points is

$$ds^{2}=c^{2}dt^{2}-dx^{2}-dy^{2}-dz^{2} \tag{3}$$

In the particle's rest frame where $t=\tau$, the action integral is therefore

$$S=-mc^{2}\int_{\tau_{1}}^{\tau_{2}}d\tau \tag{4}$$

$$=-mc\int_{\tau_{1}}^{\tau_{2}}ds \tag{5}$$

The principle of least action requires that this integral be a minimum, which in turn requires that $\int_{\tau_{1}}^{\tau_{2}}ds$ be a maximum (because of the minus sign in 5). Assuming the two points a and b are within the light cone, that is, a and b are separated by a timelike interval, which is the only possibility for a single massive particle, then $ds^{2}>0$ and ds is real, so maximizing ds^{2} means minimizing $dx^{2}+dy^{2}+dz^{2}$. Since the space components of spacetime are Euclidean, the minimum length of a path linking two points in space is a straight line, which is therefore the path followed by a free particle.

Another way of thinking about it is that if we followed some non-straight path, then every deviation from the straight line introduces some extra non-zero contributions from $dx^{2}+dy^{2}+dz^{2}$, thus increasing it and thus reducing ds^{2}.