

EULER-LAGRANGE EQUATIONS FOR A FREE PARTICLE IN AN ELECTROMAGNETIC FIELD

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Reference: Tom Lancaster and Stephen J. Blundell, *Quantum Field Theory for the Gifted Amateur*, (Oxford University Press, 2014), Problem 5.6.

Post date: 9 Mar 2019.

The treatment of a free particle in an electromagnetic field in the relativistic case closely follows that for the non-relativistic case. The difference is in the term representing the kinetic energy, which in the relativistic case includes the rest mass, so we have

$$L = -\frac{mc^2}{\gamma} + q\mathbf{A}(x) \cdot \mathbf{v} - qV(x) \quad (1)$$

where $\mathbf{A}(x)$ is the magnetic vector potential and $V(x)$ is the electric potential, both assumed to be functions of spacetime x . The particle's velocity is

$$\mathbf{v} = \dot{\mathbf{x}} \quad (2)$$

and γ is the usual relativistic factor

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} \quad (3)$$

The Euler-Lagrange equations are

$$\frac{\partial L}{\partial x_i} = \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}_i} \right) \quad (4)$$

where the index i refers to the spatial coordinates.

From 1 we have

$$\frac{\partial L}{\partial x_i} = q \frac{\partial (\mathbf{A} \cdot \mathbf{v})}{\partial x_i} - q \frac{\partial V}{\partial x_i} \quad (5)$$

We can write this as a vector equation by replacing $\partial/\partial x_i$ by ∇ and using the identity

$$\nabla (\mathbf{A} \cdot \mathbf{v}) = (\mathbf{A} \cdot \nabla) \mathbf{v} + (\mathbf{v} \cdot \nabla) \mathbf{A} + \mathbf{A} \times (\nabla \times \mathbf{v}) + \mathbf{v} \times (\nabla \times \mathbf{A}) \quad (6)$$

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Since we are taking \mathbf{x} and \mathbf{v} to be independent variables, any spatial derivatives of \mathbf{v} are zero, so this reduces to

$$\nabla(\mathbf{A} \cdot \mathbf{v}) = (\mathbf{v} \cdot \nabla)\mathbf{A} + \mathbf{v} \times (\nabla \times \mathbf{A}) \quad (7)$$

$$= (\mathbf{v} \cdot \nabla)\mathbf{A} + \mathbf{v} \times \mathbf{B} \quad (8)$$

where we used the expression for the magnetic field

$$\mathbf{B} = \nabla \times \mathbf{A} \quad (9)$$

We therefore have

$$\nabla L = q(\mathbf{v} \cdot \nabla)\mathbf{A} + q\mathbf{v} \times \mathbf{B} - q\nabla V \quad (10)$$

The RHS of 4 is

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}_i} \right) = \frac{d}{dt} \left(-\frac{\partial}{\partial v_i} \left(mc^2 \sqrt{1 - v^2/c^2} \right) + qA_i(x) \right) \quad (11)$$

We have

$$-\frac{\partial}{\partial v_i} \left(mc^2 \sqrt{1 - v^2/c^2} \right) = \frac{mv_i}{\sqrt{1 - v^2/c^2}} = \gamma mv_i \quad (12)$$

We must take the total time derivative of the last term in 11 to get

$$\frac{d}{dt} (qA_i(x)) = q \left(\frac{\partial A_i}{\partial t} + \frac{\partial A_i}{\partial x_j} \frac{dx_j}{dt} \right) \quad (13)$$

$$= q \left(\frac{\partial A_i}{\partial t} + \frac{\partial A_i}{\partial x_j} v_j \right) \quad (14)$$

where there is an implied sum over $j = 1, 2, 3$. In vector notation, we have

$$\frac{d}{dt} (q\mathbf{A}) = q \frac{\partial \mathbf{A}}{\partial t} + q(\mathbf{v} \cdot \nabla)\mathbf{A} \quad (15)$$

Inserting all this into 11 we have

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \mathbf{v}} \right) = \frac{d}{dt} (\gamma m \mathbf{v}) + q \frac{\partial \mathbf{A}}{\partial t} + q(\mathbf{v} \cdot \nabla)\mathbf{A} \quad (16)$$

Equating 10 and 16 we get, after cancelling the $q(\mathbf{v} \cdot \nabla)\mathbf{A}$ term from both sides:

$$\frac{d}{dt} (\gamma m \mathbf{v}) + q \frac{\partial \mathbf{A}}{\partial t} = q\mathbf{v} \times \mathbf{B} - q\nabla V \quad (17)$$

The electric field is given by

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$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t} \quad (18)$$

so we end up with

$$\frac{d}{dt}(\gamma m \mathbf{v}) = q \mathbf{v} \times \mathbf{B} + q \mathbf{E} \quad (19)$$

which is the Lorentz force law.

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