# ELECTROMAGNETIC LORENTZ INVARIANT

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Reference: Tom Lancaster and Stephen J. Blundell, *Quantum Field Theory for the Gifted Amateur*, (Oxford University Press, 2014), Problem 5.8.

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Since the action integral in a relativistic theory must be a Lorentz invariant, the Lagrangian must be a Lorentz scalar. For the free magnetic field, the Lagrangian is usually taken to be

$$\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} \tag{1}$$

where $F_{\mu\nu}$ is the electromagnetic field tensor

The sign convention in L&B's book is different from that used earlier.

$$F_{\mu\nu} = \begin{bmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{bmatrix} \tag{2}$$

With this sign convention, we can see by direct calculation that the components of **E** and **B** are

Latin indices $i,j,k$ represent 1, 2 and 3, while Greek indices represent 0, 1, 2, 3.

$$E_i = F_{0i} = -F_{i0} \tag{3}$$

$$B_i = -\frac{1}{2}\varepsilon_{ijk} F_{jk} \tag{4}$$

where $\varepsilon_{ijk}$ is the completely antisymmetric Levi-Civita tensor, and repeated indices are summed, as usual.

Another Lorentz invariant quantity that can be constructed from $F_{\mu\nu}$ is

$$\varepsilon^{\alpha\beta\gamma\delta} F_{\alpha\beta} F_{\gamma\delta} \tag{5}$$

where $\varepsilon^{\alpha\beta\gamma\delta}$ is the totally antisymmetric tensor where each Greek index takes the values 0, 1, 2, 3. The quantity represented by 5 is therefore a Lorentz invariant. To see what it is in terms of **E** and **B**, we can use 3. In order for a component of $\varepsilon^{\alpha\beta\gamma\delta}$ to be non-zero, all the indices must be different. Consider first the case $\alpha = 0$. The other three indices can then only be some permutation of $i,j,k$. We have for this component of the sum in 5

$$\varepsilon^{0ijk}F_{0i}F_{jk} = E_i\varepsilon^{ijk}F_{jk} \tag{6}$$
$$= -2E_iB_i \tag{7}$$
$$= -2\mathbf{E}\cdot\mathbf{B} \tag{8}$$

Now suppose $\beta = 0$. We have for this term

$$\varepsilon^{i0jk}F_{i0}F_{jk} = -\varepsilon^{0ijk}\left(-F_{0i}\right)F_{jk} \tag{9}$$
$$= \varepsilon^{0ijk}F_{0i}F_{jk} \tag{10}$$
$$= -2\mathbf{E}\cdot\mathbf{B} \tag{11}$$

In the first line, we swapped the indices $i \leftrightarrow 0$ in both $\varepsilon^{i0jk}$ and $F_{i0}$ and used the fact that both these objects are antisymmetric.

Now consider $\gamma = 0$. We have

$$\varepsilon^{ij0k}F_{ij}F_{0k} = \varepsilon^{0ijk}F_{ij}F_{0k} \tag{12}$$
$$= -2B_kE_k \tag{13}$$
$$= -2\mathbf{E}\cdot\mathbf{B} \tag{14}$$

In the first line, we did two swaps to move the 0 index in $\varepsilon^{ij0k}$ to $\varepsilon^{0ijk}$. As each swap introduces a factor of $-1$, they cancel out.

Finally, consider $\delta = 0$. We have

$$\varepsilon^{ijk0}F_{ij}F_{k0} = \varepsilon^{0ijk}F_{ij}F_{0k} \tag{15}$$
$$= -2\mathbf{E}\cdot\mathbf{B} \tag{16}$$

where in the first line, we did 3 swaps to move the 0 from $\varepsilon^{ijk0}$ to $\varepsilon^{0ijk}$, and one swap to move $F_{k0}$ to $F_{0k}$. Each of these 4 swaps introduces a factor of $-1$ so the net effect is to multiply by $(-1)^4 = 1$.

Adding everything up we have

$$\varepsilon^{\alpha\beta\gamma\delta}F_{\alpha\beta}F_{\gamma\delta} = -8\mathbf{E}\cdot\mathbf{B} \tag{17}$$

Thus $\mathbf{E}\cdot\mathbf{B}$ is a Lorentz invariant.