MAXWELL'S EQUATIONS FROM THE ELECTROMAGNETIC FIELD TENSOR

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Reference: Tom Lancaster and Stephen J. Blundell, *Quantum Field Theory for the Gifted Amateur*, (Oxford University Press, 2014), Problems 5.9-5.10.

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We've seen before how Maxwell's equations can be obtained from the electromagnetic field tensor

$$F_{\mu\nu}=\left[\begin{array}{cccc} 0 & E_{x} & E_{y} & E_{z}\\ -E_{x} & 0 & -B_{z} & B_{y}\\ -E_{y} & B_{z} & 0 & -B_{x}\\ -E_{z} & -B_{y} & B_{x} & 0 \end{array}\right] \tag{1}$$

The sign convention used in L&B is different from that used by Moore and Lahiri & Pal.

Just to review this, see this post where we show that the relation

$$\partial_{\lambda}F_{\mu\nu}+\partial_{\mu}F_{\nu\lambda}+\partial_{\nu}F_{\lambda\mu}=0 \tag{2}$$

yields two of Maxwell's equations:

$$\nabla\cdot\mathbf{B}=0 \tag{3}$$

$$\nabla\times\mathbf{E}+\frac{\partial\mathbf{B}}{\partial t}=0 \tag{4}$$

The other relation is

$$\partial_{\mu}F^{\mu\nu}=J^{\nu} \tag{5}$$

gives the other two Maxwell equations

$$\nabla\cdot\mathbf{E}=\rho \tag{6}$$

$$\nabla\times\mathbf{B}-\frac{\partial\mathbf{E}}{\partial t}=\mathbf{J} \tag{7}$$

From the facts that second derivatives are symmetric and $F_{\mu\nu}$ is antisymmetric, we have

$$\begin{aligned}\partial_\mu\partial_\nu F^{\mu\nu} &= \partial_\nu\partial_\mu F^{\mu\nu} && (8)\\ &= -\partial_\nu\partial_\mu F^{\nu\mu} && (9)\\ &= -\partial_\mu\partial_\nu F^{\mu\nu} && (10)\end{aligned}$$

In the last line, we merely swapped the indices μ and ν which is permissible since they are just dummy indices that are summed. Thus $\partial_\mu\partial_\nu F^{\mu\nu}$ is equal to its negative, so it must be zero:

$$\partial_\mu\partial_\nu F^{\mu\nu} = 0 \tag{11}$$

Thus from 5 we have

$$\partial_\nu J^\nu = 0 \tag{12}$$

Since J^ν is the four-vector

$$J^\nu = [\rho, \mathbf{J}] \tag{13}$$

and

$$\partial_\nu = \left[\frac{\partial}{\partial t}, -\nabla\right] \tag{14}$$

this gives

$$\frac{\partial\rho}{\partial t} = \nabla\cdot\mathbf{J} \tag{15}$$

which is the usual continuity equation for charge. The rate of change of charge density at a given point is equal to the divergence of the current, which carries charge into or away from that point.