KLEIN-GORDON EQUATION FROM LAGRANGIAN

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Reference: Tom Lancaster and Stephen J. Blundell, *Quantum Field Theory for the Gifted Amateur*, (Oxford University Press, 2014), Problem 6.1.

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We've seen the derivation of the Klein-Gordon equation earlier, but we'll summarize it here using notation from L&B. The Lagrangian is

$$\mathcal{L}=\frac{1}{2}\left(\partial_{\mu}\phi\right)^{2}-\frac{1}{2}m^{2}\phi^{2} \tag{1}$$

Using the Euler-Lagrange equations we have

$$\frac{\partial\mathcal{L}}{\partial\phi}=-m^{2}\phi \tag{2}$$

$$\partial_{\mu}\frac{\partial\mathcal{L}}{\partial\left(\partial_{\mu}\phi\right)}=\partial_{\mu}\partial^{\mu}\phi \tag{3}$$

Combining these gives

$$\partial_{\mu}\frac{\partial\mathcal{L}}{\partial\left(\partial_{\mu}\phi\right)}-\frac{\partial\mathcal{L}}{\partial\phi}=0 \tag{4}$$

$$\partial_{\mu}\partial^{\mu}\phi+m^{2}\phi=0 \tag{5}$$

which is the usual Klein-Gordon equation.

The momentum is

$$\pi=\frac{\partial\mathcal{L}}{\partial\dot{\phi}} \tag{6}$$

$$=\partial_{0}\phi \tag{7}$$

$$=\dot{\phi} \tag{8}$$

The Hamiltonian density is

$$\mathcal{H} = \pi\dot{\phi} - \mathcal{L} \tag{9}$$

$$= \dot{\phi}^2 - \frac{1}{2}\left(\partial_\mu\phi\right)^2 + \frac{1}{2}m^2\phi^2 \tag{10}$$

$$= \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}\left(\nabla\phi\right)^2 + \frac{1}{2}m^2\phi^2 \tag{11}$$

This is usually interpreted as the sum of a kinetic energy term $\frac{1}{2}\dot{\phi}^2$ and a potential energy term $\frac{1}{2}\left(\nabla\phi\right)^2 + \frac{1}{2}m^2\phi^2$.

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