LAGRANGIAN TO EQUATION OF MOTION: THREE EXAMPLES

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog.

Reference: Tom Lancaster and Stephen J. Blundell, *Quantum Field Theory for the Gifted Amateur*, (Oxford University Press, 2014), Problems 7.2 - 7.4.

Post date: 24 Mar 2019.

Here are three more simple examples of Lagrangians and their associated equations of motion.

Example 1. For a massive scalar field coupled to a source $J(x)$, the Lagrangian is

$$\mathcal{L}=\frac{1}{2}\left[\partial_{\mu}\phi(x)\right]^{2}-\frac{1}{2}m^{2}\left[\phi(x)\right]^{2}+J(x)\phi(x) \tag{1}$$

Using the Euler-Lagrange equations to get the equation of motion, we have

$$\partial_{\mu}\frac{\partial\mathcal{L}}{\partial(\partial_{\mu}\phi)}-\frac{\partial\mathcal{L}}{\partial\phi}=0 \tag{2}$$

So we have

$$-\frac{\partial\mathcal{L}}{\partial\phi}=m^{2}\phi(x)-J(x) \tag{3}$$

$$\partial_{\mu}\frac{\partial\mathcal{L}}{\partial(\partial_{\mu}\phi)}=\partial_{\mu}\partial^{\mu}\phi \tag{4}$$

The equation of motion is therefore

$$\left(\partial_{\mu}\partial^{\mu}\phi+m^{2}\right)\phi(x)=J(x) \tag{5}$$

Example 2. A possible Lagrangian for two interacting scalar fields is given by L&B's equation 7.14:

$$\mathcal{L}=\frac{1}{2}\left(\partial_{\mu}\phi_{1}\right)^{2}-\frac{1}{2}m^{2}\phi_{1}^{2}+\frac{1}{2}\left(\partial_{\mu}\phi_{2}\right)^{2}-\frac{1}{2}m^{2}\phi_{2}^{2}-g\left(\phi_{1}^{2}+\phi_{2}^{2}\right)^{2} \tag{6}$$

The interaction comes from the term involving $\phi_1^2\phi_2^2$ when multiplying out the last term in the Lagrangian.

Applying the Euler-Lagrange equations, we have (for $i = 1,2$)

$$-\frac{\partial \mathcal{L}}{\partial \phi_i} = m^2\phi_i + 2g\left(\phi_1^2+\phi_2^2\right)\times 2\phi_i \tag{7}$$

$$= m^2\phi_i + 4g\left(\phi_1^2+\phi_2^2\right)\phi_i \tag{8}$$

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \phi_i\right)} = \partial^\mu\partial_\mu\phi_i \tag{9}$$

We therefore have two equations of motion, one for each ϕ_i:

$$\partial^\mu\partial_\mu\phi_1 + m^2\phi_1 + 4g\left(\phi_1^2+\phi_2^2\right)\phi_1 = 0 \tag{10}$$

$$\partial^\mu\partial_\mu\phi_2 + m^2\phi_2 + 4g\left(\phi_1^2+\phi_2^2\right)\phi_2 = 0 \tag{11}$$

Example 3. The Klein-Gordon Lagrangian is given by L&B's equation 7.7:

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu\phi\right)^2 - \frac{1}{2}m^2\phi^2 \tag{12}$$

This can be written as

$$\mathcal{L} = \frac{1}{2}\partial^\mu\phi\partial_\mu\phi - \frac{1}{2}m^2\phi^2 \tag{13}$$

Using L&B's metric

$$\partial_\mu = \left(\frac{\partial}{\partial t},\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right) \tag{14}$$

Raising a spatial index introduces a minus sign, while raising the time index doesn't change anything, so

$$\partial^\mu = \left(\frac{\partial}{\partial t},-\frac{\partial}{\partial x},-\frac{\partial}{\partial y},-\frac{\partial}{\partial z}\right) \tag{15}$$

Therefore, 13 is equivalent to

$$\mathcal{L} = \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}\nabla\phi\cdot\nabla\phi - \frac{1}{2}m^2\phi^2 \tag{16}$$

$$= \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}\left(\nabla\phi\right)^2 - \frac{1}{2}m^2\phi^2 \tag{17}$$

The conjugate momentum is

$$\pi = \frac{\partial \mathcal{L}}{\partial\dot{\phi}} = \dot{\phi} \tag{18}$$

The Hamiltonian density is

$$\begin{aligned}
\mathcal{H} &= \pi\dot{\phi} - \mathcal{L} && (19)\\
&= \dot{\phi}^2 - \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2 && (20)\\
&= \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2 && (21)
\end{aligned}$$

The generalized momentum density is, using 13

$$\begin{aligned}
\Pi^\mu &= \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} && (22)\\
&= \partial^\mu\phi && (23)
\end{aligned}$$

The time component is

$$\Pi^0 = \dot{\phi} = \pi \qquad (24)$$