TIME EVOLUTION OF CREATION AND ANNIHILATON OPERATORS

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Reference: Tom Lancaster and Stephen J. Blundell, *Quantum Field Theory for the Gifted Amateur*, (Oxford University Press, 2014), Problems 8.2 - 8.3.

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The Heisenberg equation of motion for an operator Q is

$$\frac{dQ}{dt}=-\frac{i}{\hbar}\left[Q,H\right] \tag{1}$$

where H is the Hamiltonian.

For a for a real scalar field ϕ, we can write the Hamiltonian in terms of creation and annihilation operators as

$$H=\sum_{k}E_{k}a_{k}^{\dagger}a_{k} \tag{2}$$

where these operators satisfy the usual commutation relations

$$\left[a_{k},a_{j}^{\dagger}\right]=\delta_{kj} \tag{3}$$

$$\left[a_{k},a_{j}\right]=\left[a_{k}^{\dagger},a_{j}^{\dagger}\right]=0 \tag{4}$$

Applying the equation of motion we have

$$\frac{da_{k}^{\dagger}(t)}{dt}=-\frac{i}{\hbar}\left[a_{k}^{\dagger},H\right] \tag{5}$$

$$=-\frac{i}{\hbar}\sum_{j}E_{j}a_{j}^{\dagger}\left[a_{k}^{\dagger},a_{j}\right] \tag{6}$$

$$=+\frac{i}{\hbar}\sum_{j}E_{j}a_{j}^{\dagger}\left[a_{j},a_{k}^{\dagger}\right] \tag{7}$$

$$=\frac{i}{\hbar}\sum_{j}E_{j}a_{j}^{\dagger}\delta_{jk} \tag{8}$$

$$=\frac{i}{\hbar}E_{k}a_{k}^{\dagger}(t) \tag{9}$$

This differential equation has the formal solution

$$a_k^\dagger(t) = a_k^\dagger(0)\, e^{iE_k t/\hbar} \tag{10}$$

The corresponding equation for $a_k(t)$ can be obtained by a similar procedure, or by just taking the conjugate of 10:

$$a_k(t) = a_k(0)\, e^{-iE_k t/\hbar} \tag{11}$$

Now suppose we have a more general operator defined by

$$\hat{X} = X_{\ell m} a_\ell^\dagger a_m \tag{12}$$

where the $X_{\ell m}$ are just numbers (independent of time) and (I assume) there is an implied sum over ℓ and m. To find the time dependence of $\hat{X}$ we have (I'll drop the hat from X but it's understood to be an operator from here on):

$$\begin{aligned}\frac{dX}{dt} &= -\frac{i}{\hbar}[X,H] && (13)\\ &= -\frac{i}{\hbar} X_{\ell m} \sum_k E_k \left[a_\ell^\dagger a_m, a_k^\dagger a_k\right] && (14)\end{aligned}$$

We'll now work out the commutator in the sum, using 4

$$\begin{aligned}\left[a_\ell^\dagger a_m, a_k^\dagger a_k\right] &= a_\ell^\dagger a_m a_k^\dagger a_k - a_k^\dagger a_k a_\ell^\dagger a_m && (15)\\ &= a_\ell^\dagger a_k^\dagger a_m a_k + a_\ell^\dagger a_k \delta_{km} - a_k^\dagger a_\ell^\dagger a_k a_k - a_k^\dagger a_m \delta_{k\ell} && (16)\\ &= a_\ell^\dagger a_k^\dagger a_m a_k - a_\ell^\dagger a_k^\dagger a_m a_k + a_\ell^\dagger a_k \delta_{km} - a_k^\dagger a_m \delta_{k\ell} && (17)\\ &= a_\ell^\dagger a_k \delta_{km} - a_k^\dagger a_m \delta_{k\ell} && (18)\end{aligned}$$

Inserting this back into 14 we have

$$\begin{aligned}\frac{dX(t)}{dt} &= -\frac{i}{\hbar} X_{\ell m} \sum_k E_k \left(a_\ell^\dagger a_k \delta_{km} - a_k^\dagger a_m \delta_{k\ell}\right) && (19)\\ &= -\frac{i}{\hbar} X_{\ell m} \left(E_m a_\ell^\dagger a_m - E_\ell a_\ell^\dagger a_m\right) && (20)\\ &= -\frac{i}{\hbar} X_{\ell m} a_\ell^\dagger a_m (E_m - E_\ell) && (21)\\ &= -\frac{i}{\hbar} (E_m - E_\ell) X(t) && (22)\end{aligned}$$

We can again integrate this to give

$$X(t) = X(0)\, e^{i(E_\ell - E_m)t/\hbar} \tag{23}$$

We could have guessed this result by looking at 10 and 11, since in 12, the operator $a_\ell^\dagger$ evolves according to 10 and a_m evolves according to 11.