GENERATORS OF THE LORENTZ GROUP FOR 4-VECTORS

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Reference: Tom Lancaster and Stephen J. Blundell, *Quantum Field Theory for the Gifted Amateur*, (Oxford University Press, 2014), Problem 9.2.

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We've seen earlier that Lorentz transformations form a group. Although I haven't delved very deeply into group representations, one aspect of group theory is that any abstract group can have a number of representations, where each group element is represented by a square matrix. There are usually many possible representations for any given group, with different representations being given by matrices of various sizes.

For the Lorentz group, we consider boosts along each of the three coordinate axes. For a boost along the x^1 axis, the Lorentz transformation can be given as a matrix:

$$\mathbf{\Lambda}\left(\beta^1\right)=\left[\begin{array}{cccc}\gamma^1 & \beta^1\gamma^1 & 0 & 0\\ \beta^1\gamma^1 & \gamma^1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{array}\right] \tag{1}$$

where as usual, β^1 is the velocity in the x^1 direction (using natural units so that $c=1$) and

$$\gamma^1=\frac{1}{\sqrt{1-\left(\beta^1\right)^2}} \tag{2}$$

The transformation can be written using hyperbolic functions by defining the rapidity ϕ^1:

$$\mathbf{\Lambda}\left(\phi^1\right)=\left[\begin{array}{cccc}\cosh\phi^1 & \sinh\phi^1 & 0 & 0\\ \sinh\phi^1 & \cosh\phi^1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{array}\right] \tag{3}$$

At this point, L&B postulate that we can write a generalized Lorentz transformation matrix $D(\phi)$ as an exponential:

$$D(\phi)=e^{i\mathbf{K}\cdot\boldsymbol{\phi}} \tag{4}$$

where the components K^i are the generators of Lorentz boosts. From this definition, we see that

$$K^i = \frac{1}{i}\left.\frac{\partial D\left(\phi^i\right)}{\partial\phi^i}\right|_{\phi^i=0} \tag{5}$$

The form 4 appears to be just an abstract way of writing a general Lorentz transformation, without reference to any particular coordinate system or number of dimensions. For the case of 4-vectors, we can take $D\left(\phi^i\right)$ to be the same as the corresponding $\mathbf{\Lambda}\left(\phi^i\right)$ such as 3 for the x^1 direction. In this case, we have

$$\begin{aligned}
K^1 &= \frac{1}{i}\left.\frac{\partial\mathbf{\Lambda}\left(\phi^1\right)}{\partial\phi^1}\right|_{\phi^1=0} && (6)\\
&= \frac{1}{i}\frac{\partial}{\partial\phi^1}\begin{bmatrix}\cosh\phi^1 & \sinh\phi^1 & 0 & 0\\ \sinh\phi^1 & \cosh\phi^1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}_{\phi^1=0} && (7)\\
&= \frac{1}{i}\begin{bmatrix}\sinh\phi^1 & \cosh\phi^1 & 0 & 0\\ \cosh\phi^1 & \sinh\phi^1 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}_{\phi^1=0} && (8)\\
&= -i\begin{bmatrix}0 & 1 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix} && (9)
\end{aligned}$$

We can do a similar calculation for the other two coordinate directions. For x^2 (superscript '2', not 'squared'):

$$\mathbf{\Lambda}\left(\phi^2\right) = \begin{bmatrix}\cosh\phi^2 & 0 & \sinh\phi^2 & 0\\ 0 & 1 & 0 & 0\\ \sinh\phi^2 & 0 & \cosh\phi^2 & 0\\ 0 & 0 & 0 & 1\end{bmatrix} \tag{10}$$

so we get

$$K^2 = -i\begin{bmatrix}0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix} \tag{11}$$

and for x^3:

$$\mathbf{\Lambda}\left(\phi^{3}\right)=\left[\begin{array}{cccc} \cosh\phi^{3} & 0 & 0 & \sinh\phi^{3}\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ \sinh\phi^{3} & 0 & 0 & \cosh\phi^{3} \end{array}\right] \tag{12}$$

$$K^{3}=-i\left[\begin{array}{cccc} 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 1 & 0 & 0 & 0 \end{array}\right] \tag{13}$$

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