GENERAL INFINITESIMAL LORENTZ TRANSFORMATION

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Reference: Tom Lancaster and Stephen J. Blundell, *Quantum Field Theory for the Gifted Amateur*, (Oxford University Press, 2014), Problem 9.3.

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The Lorentz transformation along the x^1 axis is

$$\mathbf{\Lambda}\left(\beta^1\right)=\left[\begin{array}{cccc}\gamma^1 & \beta^1\gamma^1 & 0 & 0\\ \beta^1\gamma^1 & \gamma^1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{array}\right] \tag{1}$$

where as usual, β^1 is the velocity in the x^1 direction (using natural units so that $c=1$) and

$$\gamma^1=\frac{1}{\sqrt{1-\left(\beta^1\right)^2}} \tag{2}$$

For an infinitesimal boost $\beta^1=v^1\ll 1$, we have $\gamma^1\approx 1$ (to first order in v^1) so

$$\Lambda^\mu_{\ \nu}=\left[\begin{array}{cccc}1 & v^1 & 0 & 0\\ v^1 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{array}\right] \tag{3}$$

For an infinitesimal boost v^2 along the x^2 axis, we have

$$\Lambda^\rho_{\ \sigma}=\left[\begin{array}{cccc}1 & 0 & v^2 & 0\\ 0 & 1 & 0 & 0\\ v^2 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{array}\right] \tag{4}$$

A combined Lorentz transformation with boosts along the x^1 and x^2 axes is obtained by multiplying these two transformations (in either order; in the case of infinitesimal transformations, the matrices commute), to get

$$\Lambda^{\mu}_{\ \nu} = \begin{bmatrix} 1 & v^1 & v^2 & 0 \\ v^1 & 1 & 0 & 0 \\ v^2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{5}$$

Finally, we can add in an infintesimal boost v^3 in the x^3 direction to get

$$\Lambda^{\mu}_{\ \nu} = \begin{bmatrix} 1 & v^1 & v^2 & v^3 \\ v^1 & 1 & 0 & 0 \\ v^2 & 0 & 1 & 0 \\ v^3 & 0 & 0 & 1 \end{bmatrix} \tag{6}$$

The rotation matrix for a rotation by angle α^3 about the x^3 axis is

I think that L&B have the wrong sign for the angles of rotation.

$$R^3\left(\alpha^3\right) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha^3 & -\sin\alpha^3 & 0 \\ 0 & \sin\alpha^3 & \cos\alpha^3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{7}$$

For an infinitesimal rotation where $\alpha^3 = \theta^3 \ll 1$, this becomes

$$R^3\left(\theta^3\right) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -\theta^3 & 0 \\ 0 & \theta^3 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{8}$$

Similarly, for infinitesimal rotations about the other two axes we have

Remember that the sign for a rotation about x^2 is opposite to the other two axes.

$$R^2\left(\theta^2\right) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & \theta^2 \\ 0 & 0 & 1 & 0 \\ 0 & -\theta^2 & 0 & 1 \end{bmatrix} \tag{9}$$

$$R^1\left(\theta^1\right) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -\theta^1 \\ 0 & 0 & \theta^1 & 1 \end{bmatrix} \tag{10}$$

As with infintesimal boosts, we can multiply these matrices together in any order to get the overall rotation matrix for a combination of infintiesimal rotations about all three axes:

$$R^{\mu}_{\ \nu}=\begin{bmatrix}1&0&0&0\\0&1&-\theta^3&\theta^2\\0&\theta^3&1&-\theta^1\\0&-\theta^2&\theta^1&1\end{bmatrix} \tag{11}$$

Finally, if we multiply 6 by 11 and keep only first order terms (so we ignore terms like $v^1\theta^2$ since they are products of two infinitesimals), we get the general Lorentz transformation for a combination of infinitesimal boosts and rotations.

$$\Lambda^{\mu}_{\ \nu}=\begin{bmatrix}1&v^1&v^2&v^3\\v^1&1&-\theta^3&\theta^2\\v^2&\theta^3&1&-\theta^1\\v^3&-\theta^2&\theta^1&1\end{bmatrix} \tag{12}$$

We can write this as $\Lambda=I+\boldsymbol{\omega}$ where the infinitesimal matrix is

$$\omega^{\mu}_{\ \nu}=\begin{bmatrix}0&v^1&v^2&v^3\\v^1&0&-\theta^3&\theta^2\\v^2&\theta^3&0&-\theta^1\\v^3&-\theta^2&\theta^1&0\end{bmatrix} \tag{13}$$

We can raise or lower indices on $\omega^{\mu}_{\ \nu}$ by using the metric tensor $g_{\mu\nu}$, where $g_{00}=g^{00}=+1$ and $g_{ii}=g^{ii}=-1$. Thus we have

$$\omega^{\mu\nu}=\omega^{\mu}_{\ \lambda}g^{\lambda\nu} \tag{14}$$

$$=\begin{bmatrix}0&-v^1&-v^2&-v^3\\v^1&0&\theta^3&-\theta^2\\v^2&-\theta^3&0&\theta^1\\v^3&\theta^2&-\theta^1&0\end{bmatrix} \tag{15}$$

$$\omega_{\mu\nu}=g_{\mu\lambda}\omega^{\lambda}_{\ \nu} \tag{16}$$

$$=\begin{bmatrix}0&v^1&v^2&v^3\\-v^1&0&\theta^3&-\theta^2\\-v^2&-\theta^3&0&\theta^1\\-v^3&\theta^2&-\theta^1&0\end{bmatrix} \tag{17}$$

We can see by inspection that both $\omega^{\mu\nu}$ and $\omega_{\mu\nu}$ are antisymmetric, and that

Again, I think L&B have the sign wrong in their equation 9.58.

$$v^i=-\omega^{0i}=\omega_{0i} \tag{18}$$

$$\theta^i=\frac{1}{2}\varepsilon^{ijk}\omega^{jk} \tag{19}$$

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