POINCARÉ GROUP TRANSFORMATIONS

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Reference: Tom Lancaster and Stephen J. Blundell, *Quantum Field Theory for the Gifted Amateur*, (Oxford University Press, 2014), Problem 9.4.

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[My solution of this problem differs in some of the signs from that stated in the textbook. Although the errata page given by the authors does show some corrections to the textbook, I don't think these corrections are, in fact, correct. Comments welcome.]

The Poincaré group is the combination of the Lorentz group (with boosts and rotations) with translations. Thus an infinitesimal transformation is given by

$$x'^{\mu}=x^{\mu}+a^{\mu}+\omega_{\ \nu}^{\mu}x^{\nu} \tag{1}$$

where a^{μ} is the translation and the matrix $\omega_{\ \nu}^{\mu}$ represents the Lorentz transformation

$$\omega_{\ \nu}^{\mu}=\left[\begin{array}{cccc}
0 & v^{1} & v^{2} & v^{3}\\
v^{1} & 0 & -\theta^{3} & \theta^{2}\\
v^{2} & \theta^{3} & 0 & -\theta^{1}\\
v^{3} & -\theta^{2} & \theta^{1} & 0
\end{array}\right] \tag{2}$$

Using a Taylor expansion, a function $f(x)$ therefore transforms as follows:

$$f\left(x'\right)=f(x)+a^{\mu}\partial_{\mu}f(x)+\omega_{\ \nu}^{\mu}x^{\nu}\partial_{\mu}f(x) \tag{3}$$

By lowering the μ index on $\omega_{\ \nu}^{\mu}$ we can rewrite this as

$$f\left(x'\right)=f(x)+a^{\mu}\partial_{\mu}f(x)-\omega_{\mu\nu}x^{\nu}\partial^{\mu}f(x) \tag{4}$$

Since

$$\omega_{\mu\nu}=\left[\begin{array}{cccc}
0 & v^{1} & v^{2} & v^{3}\\
-v^{1} & 0 & \theta^{3} & -\theta^{2}\\
-v^{2} & -\theta^{3} & 0 & \theta^{1}\\
-v^{3} & \theta^{2} & -\theta^{1} & 0
\end{array}\right] \tag{5}$$

the matrix is antisymmetric, so $\omega_{\mu\nu}=-\omega_{\nu\mu}$. We can therefore rewrite 4 as

$$f\left(x'\right) = f(x) + a^{\mu}\partial_{\mu}f(x) + \frac{1}{2}\left(\omega_{\mu\nu}x^{\nu}\partial^{\mu} + \omega_{\nu\mu}x^{\mu}\partial^{\nu}\right)f(x) \tag{6}$$
$$= f(x) + a^{\mu}\partial_{\mu}f(x) + \frac{1}{2}\left(\omega_{\mu\nu}x^{\nu}\partial^{\mu} - \omega_{\mu\nu}x^{\mu}\partial^{\nu}\right)f(x) \tag{7}$$
$$= f(x) + a^{\mu}\partial_{\mu}f(x) + \frac{1}{2}\omega_{\mu\nu}\left(x^{\nu}\partial^{\mu} - x^{\mu}\partial^{\nu}\right)f(x) \tag{8}$$
$$= f(x) + a^{\mu}\partial_{\mu}f(x) - \frac{1}{2}\omega_{\mu\nu}\left(x^{\mu}\partial^{\nu} - x^{\nu}\partial^{\mu}\right)f(x) \tag{9}$$

By defining

$$p_{\mu} = -i\partial_{\mu} \tag{10}$$
$$M^{\mu\nu} = -i\left(x^{\mu}\partial^{\nu} - x^{\nu}\partial^{\mu}\right) \tag{11}$$

we have finally

$$f\left(x'\right) == \left[1 + ia^{\mu}p_{\mu} - \frac{i}{2}\omega_{\mu\nu}M^{\mu\nu}\right]f(x) \tag{12}$$

At this point, L&B relate the quantity $M^{\mu\nu}$ to the Lorentz generators J^{i} and K^{i}.

$$J^{i} = \frac{1}{2}\varepsilon^{ijk}M^{jk} \tag{13}$$
$$K^{i} = M^{0i} \tag{14}$$

We can now try to derive the final result in the problem, which is to show that

$$\mathbf{\Lambda} = \exp\left(-\frac{i}{2}\omega_{\mu\nu}M^{\mu\nu}\right) \tag{15}$$

where

$$\mathbf{\Lambda} = e^{-i(\mathbf{J}\cdot\boldsymbol{\theta} - \mathbf{K}\cdot\boldsymbol{\phi})} \tag{16}$$

We thus need to show that

$$\mathbf{J}\cdot\boldsymbol{\theta} = \frac{1}{2}\omega_{ij}M^{ij} \tag{17}$$
$$\mathbf{K}\cdot\boldsymbol{\phi} = -\frac{1}{2}\left(\omega_{0i}M^{0i} + \omega_{i0}M^{i0}\right) \tag{18}$$

For the first case, we can use the earlier result

$$\theta^{i}=\frac{1}{2}\varepsilon^{ijk}\omega_{jk} \tag{19}$$

Substituting this and 13 into 17 we have (using the antisymmetry of both $\omega_{\mu\nu}$ and $M^{\mu\nu}$):

$$\mathbf{J}\cdot\boldsymbol{\theta}=\left(\frac{1}{2}\varepsilon^{ijk}M^{jk}\right)\left(\frac{1}{2}\varepsilon^{ilm}\omega_{lm}\right) \tag{20}$$

$$=\frac{1}{4}\left[\left(M^{23}-M^{32}\right)\left(\omega_{23}-\omega_{32}\right)+\left(M^{12}-M^{21}\right)\left(\omega_{12}-\omega_{21}\right)+\left(M^{31}-M^{13}\right)\left(\omega_{31}-\omega_{13}\right)\right] \tag{21}$$

$$=\frac{1}{4}\left[2M^{23}\times2\omega_{23}+2M^{12}\times2\omega_{12}+2M^{31}\times2\omega_{31}\right] \tag{22}$$

$$=\frac{1}{2}\omega_{jk}M^{jk} \tag{23}$$

where to get the factor of $\frac{1}{2}$ in the last line, we used $M^{23}\omega_{23}=\frac{1}{2}\left(M^{23}\omega_{23}+M^{32}\omega_{32}\right)$ and so on.

For 18, we assume that ϕ is the rapidity which, for infinitesimals, is the same as the velocity $v^{i}=\omega_{0i}$. Therefore

$$\mathbf{K}\cdot\boldsymbol{\phi}=\omega_{0i}M^{0i} \tag{24}$$

$$=\frac{1}{2}\left(\omega_{0i}M^{0i}+\omega_{i0}M^{i0}\right) \tag{25}$$

Unfortunately, this is the wrong sign when compared with 18. We do get the right sign if we accept L&B's equation 9.58, which says that

$$v^{i}=\omega^{0i}=-\omega_{0i} \tag{26}$$

and

$$\theta^{i}=-\frac{1}{2}\varepsilon^{ijk}\omega_{jk} \tag{27}$$

but given their form of the original matrix ω_{ν}^{μ} where all the v^{i} entries are positive, that cannot be correct.

In the errata for the book, L&B say that their version of 4 is correct, which is

$$f\left(x'\right)=f\left(x\right)-a^{\mu}\partial_{\mu}f\left(x\right)+\omega_{\mu\nu}x^{\nu}\partial^{\mu}f\left(x\right) \tag{28}$$

but that the signs for p_{μ} and $M^{\mu\nu}$ should be reversed, so we have

$$p_\mu = i\partial_\mu \tag{29}$$
$$M^{\mu\nu} = i\left(x^\mu \partial^\nu - x^\nu \partial^\mu\right) \tag{30}$$

If we take 27 to be correct, then the signs of the angles in 2 (and in the subsequent versions $\omega_{\mu\nu}$ and $\omega^{\mu\nu}$) are reversed, which seems to be the wrong way round for rotation matrices. However, if we accept both 27 and 30, then both M^{jk} and ω_{lm} change sign in 23, but if we accept both 26 and 30, then both the ωs and Ms change sign in 25, leaving us with a $\mathbf{K}\cdot\boldsymbol{\phi}$ that still has the wrong sign.

Clearly something is muddled with the signs somewhere in all this, but I'm not sure where. Comments welcome.