ENERGY-MOMENTUM TENSOR: COMMUTATOR OF FIELD WITH CONSERVED CHARGE

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Reference: Tom Lancaster and Stephen J. Blundell, *Quantum Field Theory for the Gifted Amateur*, (Oxford University Press, 2014), Problem 10.1.

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We've seen before the derivation of the energy-momentum tensor from Noether's theorem. L&B derive the tensor using a slightly different argument from those we've seen before, but I won't copy it all out again here, as the result is the same. In L&B's notation, we have

$$T^{\mu\nu}=\Pi^{\mu}\partial^{\nu}\phi-g^{\mu\nu}\mathcal{L} \tag{1}$$

where $\mathcal{L}$ is the Lagrangian density, ϕ is the field and the canonical momentum density is

$$\Pi^{\mu}\equiv\frac{\partial\mathcal{L}}{\partial\left(\partial_{\mu}\phi\right)} \tag{2}$$

The conserved charges from Noether's theorem are

$$P^{\alpha}=\int d^{3}x\,T^{0\alpha} \tag{3}$$

In their problem 10.1, L&B ask us to show

$$\left[\phi\left(x\right),P^{\alpha}\right]=i\partial^{\alpha}\phi\left(x\right) \tag{4}$$

It's not clear to me how we can do this using only material that has been presented in L&B's book up to this point. As far as I can tell, the presentation in Chapter 10 refers to classical field theory, in which commutators don't have any meaning. We *have* seen a derivation of 4 before (here) but this relies on Fourier expansions of ϕ and Π in terms of creation and annihilation operators, which haven't yet been introduced in L&B.

However, if we assume the usual quantum commutation relation between the field and the conjugate momentum $\pi\left(x\right)$

$$\pi\left(x\right)=\frac{\partial\mathcal{L}}{\partial\dot{\phi}}=\Pi^{0}\left(x\right) \tag{5}$$

which is

$$\left[\phi\left(x\right),\pi\left(y\right)\right]=i\delta^{(3)}\left(\mathbf{x}-\mathbf{y}\right) \tag{6}$$

then we can write 3 as (changing the variable of integration from x to y for convenience later):

$$P^{\alpha}=\int d^{3}y\ \left(\pi\left(y\right)\partial^{\alpha}\phi\left(y\right)-g^{\mu\nu}\mathcal{L}\left(y\right)\right) \tag{7}$$

and thus get (assuming that the field commutes with the Lagrangian density $\mathcal{L}$):

$$\begin{aligned}\left[\phi\left(x\right),P^{\alpha}\right] &= \int d^{3}y\ \left[\phi\left(x\right),\pi\left(y\right)\right]\partial^{\alpha}\phi\left(y\right) && (8)\\ &= \int d^{3}y\ i\delta^{(3)}\left(\mathbf{x}-\mathbf{y}\right)\partial^{\alpha}\phi\left(y\right) && (9)\\ &= \partial^{\alpha}\phi\left(x\right) && (10)\end{aligned}$$

We can take $\phi\left(x\right)$ inside the integral since its spacetime variable x is not integrated over.

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