NOETHER CURRENT FOR SYSTEM OF MANY SCALAR FIELDS

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Reference: Tom Lancaster and Stephen J. Blundell, *Quantum Field Theory for the Gifted Amateur*, (Oxford University Press, 2014), Problem 10.2.

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L&B derive the Noether current J^{μ} for a Lagrangian that depends on a single scalar field $\phi(x)$ in their Chapter 10. Problem 10.2 asks us to go through the same derivation for a Lagrangian that depends on N scalar fields $\phi_1,\ldots,\phi_N$ so that

$$\mathcal{L}=\mathcal{L}\left(\phi_1,\ldots,\phi_N;\partial_\mu\phi_1,\ldots,\partial_\mu\phi_N;x^\mu\right) \tag{1}$$

The derivation is very similar to that given in the book. The main difference is that we need to use the chain rule for derivatives in a few places. We'll run through the derivation here. We get (where repeated indices are summed, with a being summed from 1 to N):

$$\delta\mathcal{L}=\frac{\partial\mathcal{L}}{\partial\phi^a}\delta\phi^a+\frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\phi^a\right)}\delta\left(\partial_\mu\phi^a\right) \tag{2}$$

$$=\frac{\partial\mathcal{L}}{\partial\phi^a}\delta\phi^a+\Pi_a^\mu\delta\left(\partial_\mu\phi^a\right) \tag{3}$$

where

$$\Pi_a^\mu\equiv\frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\phi^a\right)} \tag{4}$$

We now use the product rule to write

$$\partial_\mu\left(\Pi_a^\mu\delta\phi^a\right)=\Pi_a^\mu\delta\left(\partial_\mu\phi^a\right)+\partial_\mu\left(\Pi_a^\mu\right)\delta\phi^a \tag{5}$$

Using this, we rewrite 3 as

$$\delta\mathcal{L}=\frac{\partial\mathcal{L}}{\partial\phi^a}\delta\phi^a+\partial_\mu\left(\Pi_a^\mu\delta\phi^a\right)-\partial_\mu\left(\Pi_a^\mu\right)\delta\phi^a \tag{6}$$

$$=\left(\frac{\partial\mathcal{L}}{\partial\phi^a}-\partial_\mu\left(\Pi_a^\mu\right)\right)\delta\phi^a+\partial_\mu\left(\Pi_a^\mu\delta\phi^a\right) \tag{7}$$

We now require the system to satisfy the Euler-Lagrange equations of motion, which state

$$\frac{\partial\mathcal{L}}{\partial\phi^{a}}-\partial_{\mu}\left(\Pi_{a}^{\mu}\right)=0 \tag{8}$$

so that we have

$$\delta\mathcal{L}=\partial_{\mu}\left(\Pi_{a}^{\mu}\delta\phi^{a}\right) \tag{9}$$

We now consider a translation along a spacetime vector b^{μ} by an amount λb^{μ}, where λ is some parameter that can be varied to vary the size of the translation. Under this transformation the N fields will transform as

$$\phi_{a}\left(x^{\mu}\right)\rightarrow\phi_{a}\left(x^{\mu}+\lambda b^{\mu}\right) \tag{10}$$

We then define the quantity

$$D\phi^{a}\equiv\left.\frac{\partial\phi^{a}}{\partial\lambda}\right|_{\lambda=0}=b^{\mu}\partial_{\mu}\phi^{a} \tag{11}$$

so that

$$\delta\phi^{a}=D\phi^{a}\delta\lambda \tag{12}$$

The goal is that the action

$$S=\int d^{4}x\;\mathcal{L} \tag{13}$$

remain stationary under the symmetry transformation (since we're assuming that the system is invariant under this symmetry transformation). One way of ensuring this is to require $\delta\mathcal{L}=0$, but since the change in action is

$$\delta S=\int d^{4}x\;\delta\mathcal{L} \tag{14}$$

where the integral extends over all spacetime, we can allow $\delta\mathcal{L}$ to be some total divergence, provided that we can apply the divergence theorem to convert the spacetime integral to a surface integral, which as usual vanishes as we go to infinity. That is, we can let

$$\delta\mathcal{L}=\left(\partial_{\mu}W^{\mu}\right)\delta\lambda \tag{15}$$

for some function $W^{\mu}\left(x\right)$. For a given W^{μ}, the effect this has on the action integral 13 will be a change in the total action of some constant value (which could be zero), so if $\delta S=0$ when $\delta\mathcal{L}=0$ it will still be zero if we add on the term $\left(\partial_{\mu}W^{\mu}\right)\delta\lambda$ since we're not varying any of the fields in doing this.

Equating this with 9 we get the condition

$$\partial_\mu \left(\Pi_a^\mu \delta\phi^a - \left(\partial_\mu W^\mu\right)\delta\lambda\right) = \partial_\mu \left(\Pi_a^\mu D\phi^a - \partial_\mu W^\mu\right)\delta\lambda \tag{16}$$

$$= 0 \tag{17}$$

which gives the Noether current

$$J_N^\mu\left(x\right) = \Pi_a^\mu D\phi^a - \partial_\mu W^\mu \tag{18}$$

which satisfies the conservation rule

$$\partial_\mu J_N^\mu\left(x\right) = 0 \tag{19}$$