ENERGY-MOMENTUM TENSOR FOR KLEIN-GORDON LAGRANGIAN

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Reference: Tom Lancaster and Stephen J. Blundell, *Quantum Field Theory for the Gifted Amateur*, (Oxford University Press, 2014), Problem 10.3.

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Here we'll investigate the energy-momentum tensor for the Klein-Gordon Lagrangian. The Lagrangian is

$$\mathcal{L}=\frac{1}{2}\left(\partial_{\mu}\phi\right)^{2}-\frac{1}{2}m^{2}\phi^{2} \tag{1}$$

and the energy-momentum tensor is

$$T^{\mu\nu}=\Pi^{\mu}\partial^{\nu}\phi-g^{\mu\nu}\mathcal{L} \tag{2}$$

where $\mathcal{L}$ is the Lagrangian density, ϕ is the field and the canonical momentum density is

$$\Pi^{\mu}\equiv\frac{\partial\mathcal{L}}{\partial\left(\partial_{\mu}\phi\right)} \tag{3}$$

We have

$$T^{\mu\nu}=\partial^{\mu}\phi\partial^{\nu}\phi-g^{\mu\nu}\mathcal{L} \tag{4}$$

The energy component is

$$T^{00}=\dot{\phi}^{2}-\mathcal{L} \tag{5}$$

$$=\frac{1}{2}\left(\dot{\phi}^{2}+(\nabla\phi)^{2}+m^{2}\phi^{2}\right) \tag{6}$$

where we've used

$$\left(\partial_{\mu}\phi\right)^{2}=\partial_{\mu}\phi\partial^{\mu}\phi \tag{7}$$

$$=\dot{\phi}^{2}-\nabla\phi\cdot\nabla\phi \tag{8}$$

$$=\dot{\phi}^{2}-(\nabla\phi)^{2} \tag{9}$$

We see that 6 agrees with the earlier expression for the Hamiltonian density.

One condition that any energy-momentum tensor must satisfy in order to conserve the Noether current is

$$\partial_\mu T^{\mu\nu} = 0 \tag{10}$$

We can see this explicitly for this case by calculating derivatives.

$$\begin{aligned}
\partial_\mu T^{\mu\nu} &= (\partial_\mu \partial^\mu \phi)(\partial^\nu \phi) + (\partial^\mu \phi)(\partial_\mu \partial^\nu \phi) - g^{\mu\nu}\left(\frac{1}{2}\partial_\mu (\partial^\rho \phi \partial_\rho \phi) - m^2 \phi \partial_\mu \phi\right) && (11)\\
&= (\partial_\mu \partial^\mu \phi)(\partial^\nu \phi) + (\partial^\mu \phi)(\partial_\mu \partial^\nu \phi) - g^{\mu\nu}\left((\partial_\mu \partial^\rho \phi)\partial_\rho \phi - m^2 \phi \partial_\mu \phi\right) && (12)\\
&= (\partial_\mu \partial^\mu \phi)(\partial^\nu \phi) + (\partial^\mu \phi)(\partial_\mu \partial^\nu \phi) - (\partial^\nu \partial^\rho \phi)\partial_\rho \phi + m^2 \phi \partial^\nu \phi && (13)
\end{aligned}$$

where in the last line we used $g^{\mu\nu}$ to raise the μ index in each of the last two terms.

To simplify this we make use of the Euler-Lagrange equation for this system, which is

$$\frac{\partial \mathcal{L}}{\partial \phi} = \partial_\mu \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \tag{14}$$

In this case, this equation becomes

$$-m^2 \phi = \partial^\mu \partial_\mu \phi \tag{15}$$

we we get

$$\begin{aligned}
\partial_\mu T^{\mu\nu} &= -m^2 \phi (\partial^\nu \phi) + (\partial^\mu \phi)(\partial_\mu \partial^\nu \phi) - (\partial^\nu \partial^\rho \phi)\partial_\rho \phi + m^2 \phi \partial^\nu \phi && (16)\\
&= -m^2 \phi (\partial^\nu \phi) + (\partial^\mu \phi)(\partial_\mu \partial^\nu \phi) - (\partial^\nu \partial^\mu \phi)\partial_\mu \phi + m^2 \phi \partial^\nu \phi && (17)\\
&= 0 && (18)
\end{aligned}$$

where in the second line, we relabelled the dummy (summed) index ρ to μ.

The conserved charges are, from 4

$$P^0 = \int d^3x\, T^{00} \tag{19}$$

$$= \frac{1}{2}\int d^3x\, \left(\dot{\phi}^2 + (\nabla\phi)^2 + m^2\phi^2\right) \tag{20}$$

$$P^i = \int d^3x\, T^{0i} \tag{21}$$

$$= \int d^3x\, \dot{\phi}\partial^i\phi \tag{22}$$

$$= \int d^3x\, \pi(x)\, \partial^i\phi(x) \tag{23}$$

where $\pi(x)$ is the conjugate momentum density.