ENERGY-MOMENTUM TENSOR FOR ELECTROMAGNETISM

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog.

Reference: Tom Lancaster and Stephen J. Blundell, *Quantum Field Theory for the Gifted Amateur*, (Oxford University Press, 2014), Problem 10.4.

Post date: 16 Apr 2019.

The electromagnetic field tensor used by L&B is

$$F_{\mu\nu}=\begin{bmatrix}0 & E_x & E_y & E_z\\ -E_x & 0 & -B_z & B_y\\ -E_y & B_z & 0 & -B_x\\ -E_z & -B_y & B_x & 0\end{bmatrix}\tag{1}$$

$$F^{\mu\nu}=\begin{bmatrix}0 & -E_x & -E_y & -E_z\\ E_x & 0 & -B_z & B_y\\ E_y & B_z & 0 & -B_x\\ E_z & -B_y & B_x & 0\end{bmatrix}\tag{2}$$

The free field electromagnetic Lagrangian is

$$\mathcal{L}=-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}=\frac{1}{2}\left(\mathbf{E}^2-\mathbf{B}^2\right)\tag{3}$$

where the last equality is obtained by substituting for $F_{\mu\nu}$ from 1.

We're asked to show that

$$\Pi^{\sigma\rho}\equiv\frac{\partial\mathcal{L}}{\partial\left(\partial_\sigma A_\rho\right)}=-F^{\sigma\rho}\tag{4}$$

For this, we use the standard definition of $F^{\mu\nu}$ in terms of the four-potential A^μ:

$$F^{\mu\nu}=\partial^\mu A^\nu-\partial^\nu A^\mu\tag{5}$$

When working out things like 4, there is probably a quick and easy way of doing it, but the mixture of upper and lower indices tends to confuse me, so in order to make sure I get it right, I expand things using the metric tensor. Using this notation, we have

$$\mathcal{L} = -\frac{1}{4}g^{\mu\alpha}g^{\nu\beta}F_{\alpha\beta}F_{\mu\nu} \tag{6}$$

$$= -\frac{1}{4}g^{\mu\alpha}g^{\nu\beta}\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)\left(\partial_\alpha A_\beta - \partial_\beta A_\alpha\right) \tag{7}$$

Using 5 and the product rule, we can now work out 4. We have

$$\frac{\partial\mathcal{L}}{\partial\left(\partial_\sigma A_\rho\right)} = -\frac{1}{4}g^{\mu\alpha}g^{\nu\beta}\left(\delta_{\sigma\mu}\delta_{\rho\nu} - \delta_{\sigma\nu}\delta_{\rho\mu}\right)\left(\partial_\alpha A_\beta - \partial_\beta A_\alpha\right)$$
$$-\frac{1}{4}g^{\mu\alpha}g^{\nu\beta}\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)\left(\delta_{\sigma\alpha}\delta_{\rho\beta} - \delta_{\sigma\beta}\delta_{\rho\alpha}\right) \tag{8}$$

$$= -\frac{1}{4}\left(g^{\sigma\alpha}g^{\rho\beta} - g^{\rho\alpha}g^{\sigma\beta}\right)\left(\partial_\alpha A_\beta - \partial_\beta A_\alpha\right)$$
$$-\frac{1}{4}\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)\left(g^{\mu\sigma}g^{\nu\rho} - g^{\mu\rho}g^{\nu\sigma}\right) \tag{9}$$

$$= -\frac{1}{4}\left(\partial^\sigma A^\rho - \partial^\rho A^\sigma - \partial^\rho A^\sigma + \partial^\sigma A^\rho\right)$$
$$-\frac{1}{4}\left(\partial^\sigma A^\rho - \partial^\rho A^\sigma - \partial^\rho A^\sigma + \partial^\sigma A^\rho\right) \tag{10}$$

$$= -\left(\partial^\sigma A^\rho - \partial^\rho A^\sigma\right) \tag{11}$$

$$= -F^{\sigma\rho} \tag{12}$$

The energy-momentum tensor for this Lagrangian is then

$$T^\mu_\nu = \Pi^{\mu\sigma}\partial_\nu A_\sigma - \delta^\mu_\nu\mathcal{L} \tag{13}$$

$$= -F^{\mu\sigma}\partial_\nu A_\sigma - \delta^\mu_\nu\mathcal{L} \tag{14}$$

Raising the ν index gives

$$T^{\mu\nu} = -F^{\mu\sigma}\partial^\nu A_\sigma - g^{\mu\nu}\mathcal{L} \tag{15}$$

$$= -F^{\mu\sigma}\partial^\nu A_\sigma + \frac{1}{4}g^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} \tag{16}$$

We can symmetrize $T^{\mu\nu}$ by adding $\partial_\lambda X^{\lambda\mu\nu}$ where

$$X^{\lambda\mu\nu} = F^{\mu\lambda}A^\nu \tag{17}$$

This quantity is antisymmetric in its first two indices, as we can verify:

$$X^{\lambda\mu\nu} = \left(\partial^{\mu}A^{\lambda} - \partial^{\lambda}A^{\mu}\right)A^{\nu} \tag{18}$$

$$= -\left(\partial^{\lambda}A^{\mu} - \partial^{\mu}A^{\lambda}\right)A^{\nu} \tag{19}$$

$$= -X^{\mu\lambda\nu} \tag{20}$$

The derivative is

$$\partial_{\lambda}X^{\lambda\mu\nu} = \partial_{\lambda}F^{\mu\lambda}A^{\nu} + F^{\mu\lambda}\partial_{\lambda}A^{\nu} \tag{21}$$

When we looked at Maxwell's equations in terms of the field tensor, we had the equation

$$\partial_{\lambda}F^{\mu\lambda} = j^{\mu} \tag{22}$$

where j^{μ} is the four-current. For a free field, $j^{\mu} = 0$ so in our case

$$\partial_{\lambda}F^{\mu\lambda} = 0 \tag{23}$$

$$\partial_{\lambda}X^{\lambda\mu\nu} = F^{\mu\lambda}\partial_{\lambda}A^{\nu} \tag{24}$$

The modified energy-momentum tensor $\tilde{T}^{\mu\nu}$ is then

$$\tilde{T}^{\mu\nu} = T^{\mu\nu} + \partial_{\sigma}X^{\sigma\mu\nu} \tag{25}$$

$$= -F^{\mu\sigma}\partial^{\nu}A_{\sigma} + \frac{1}{4}g^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} + F^{\mu\sigma}\partial_{\sigma}A^{\nu} \tag{26}$$

$$= F^{\mu\sigma}\left(\partial_{\sigma}A^{\nu} - \partial^{\nu}A_{\sigma}\right) + \frac{1}{4}g^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} \tag{27}$$

$$= F^{\mu\sigma}F_{\sigma}^{\ \nu} + \frac{1}{4}g^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} \tag{28}$$

To verify that $\tilde{T}^{\mu\nu}$ is actually symmetric, we have, repeating 2 and then lowering the first index:

$$F^{\mu\sigma} = \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{bmatrix} \tag{29}$$

$$F_{\sigma}^{\ \nu} = \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{bmatrix} \tag{30}$$

We can see by direct calculation that $F^{\mu\sigma}F_{\sigma}^{\ \nu}=F^{\nu\sigma}F_{\sigma}^{\ \mu}$. The matrix product gives

$$F^{\mu\sigma}F_{\sigma}^{\ \nu}=\left[\begin{array}{cccc} E_{x}^{2}+E_{y}^{2}+E_{z}^{2} & -E_{z}B_{y}+E_{y}B_{z} & E_{z}B_{x}-E_{x}B_{z} & -E_{y}B_{x}+E_{x}B_{y}\\ -E_{z}B_{y}+E_{y}B_{z} & B_{y}^{2}+B_{z}^{2}-E_{x}^{2} & -B_{y}B_{x}-E_{x}E_{y} & -B_{z}B_{x}-E_{x}E_{z}\\ E_{z}B_{x}-E_{x}B_{z} & -B_{y}B_{x}-E_{x}E_{y} & B_{x}^{2}+B_{z}^{2}-E_{y}^{2} & -B_{z}B_{y}-E_{y}E_{z}\\ -E_{y}B_{x}+E_{x}B_{y} & -B_{z}B_{x}-E_{x}E_{z} & -B_{z}B_{y}-E_{y}E_{z} & B_{x}^{2}+B_{y}^{2}-E_{z}^{2} \end{array}\right] \tag{31}$$

For individual components of $\tilde{T}^{\mu\nu}$ we have, using 3

$$\tilde{T}^{00}=F^{0\lambda}F_{\lambda}^{\ 0}+\frac{1}{4}F_{\alpha\beta}F^{\alpha\beta} \tag{32}$$

$$=(-\mathbf{E})\cdot(-\mathbf{E})-\frac{1}{2}\left(\mathbf{E}^{2}-\mathbf{B}^{2}\right) \tag{33}$$

$$=\frac{1}{2}\left(\mathbf{E}^{2}+\mathbf{B}^{2}\right) \tag{34}$$

For elements $\tilde{T}^{i0}$ we have

$$\tilde{T}^{i0}=F^{i\sigma}F_{\sigma}^{\ 0} \tag{35}$$

which we can read off from column 0 in 31. We see that, for example

$$\tilde{T}^{10}=-E_{z}B_{y}+E_{y}B_{z} \tag{36}$$

$$=(\mathbf{E}\times\mathbf{B})_{1} \tag{37}$$

so in general

$$\tilde{T}^{i0}=(\mathbf{E}\times\mathbf{B})_{i} \tag{38}$$

PINGBACKS

Pingback: Massive electromagnetism: Hamiltonian density