COMMUTATOR OF REAL SCALAR FIELDS AT GENERAL TIMES

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Reference: Tom Lancaster and Stephen J. Blundell, *Quantum Field Theory for the Gifted Amateur*, (Oxford University Press, 2014), Problem 11.1.

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The real scalar field $\phi(x)$ can be written as a Fourier expansion

$$\phi(x)=\frac{1}{(2\pi)^{3/2}}\int\frac{d^3p}{\sqrt{2E_p}}\left(a(p)\,e^{-ip\cdot x}+a^{\dagger}(p)\,e^{ip\cdot x}\right) \tag{1}$$

where the energy is

$$E_{\mathbf{p}}=+\sqrt{\mathbf{p}^2+m^2} \tag{2}$$

The field operators and their conjugate momenta $\Pi(x)$ are postulated to have *equal time commutators*

$$[\phi(t,\mathbf{x}),\Pi(t,\mathbf{y})]=i\delta^3(\mathbf{x}-\mathbf{y}) \tag{3}$$

$$[\phi(t,\mathbf{x}),\phi(t,\mathbf{y})]=0 \tag{4}$$

$$[\Pi(t,\mathbf{x}),\Pi(t,\mathbf{y})]=0 \tag{5}$$

from which we can derive the commutators of the creation and annihilation operators as

$$\left[a(p),a^{\dagger}(q)\right]=\delta^{(3)}(\mathbf{p}-\mathbf{q}) \tag{6}$$

$$[a(p),a(q)]=\left[a^{\dagger}(p),a^{\dagger}(q)\right]=0 \tag{7}$$

We can use these commutators to find the commutator $[\phi(x),\phi(y)]$ where now the time components of x and y are not necessarily the same. From 1 we have

$$\left[\phi(x),\phi(y)\right]=\int\frac{d^3p\,d^3q}{2(2\pi)^3\sqrt{E_{\mathbf{p}}E_{\mathbf{q}}}}\left[a(p)\,e^{-ip\cdot x}+a^\dagger(p)\,e^{ip\cdot x},a(q)\,e^{-iq\cdot y}+a^\dagger(q)\,e^{iq\cdot y}\right] \tag{8}$$

$$=\int\frac{d^3p\,d^3q}{2(2\pi)^3\sqrt{E_{\mathbf{p}}E_{\mathbf{q}}}}\left\{\left[a(p),a^\dagger(q)\right]e^{-i(p\cdot x-q\cdot y)}+\left[a^\dagger(p),a(q)\right]e^{i(p\cdot x-q\cdot y)}\right\} \tag{9}$$

$$=\int\frac{d^3p\,d^3q}{2(2\pi)^3\sqrt{E_{\mathbf{p}}E_{\mathbf{q}}}}\left\{\delta^{(3)}(\mathbf{p}-\mathbf{q})\,e^{-i(p\cdot x-q\cdot y)}-\delta^{(3)}(\mathbf{p}-\mathbf{q})\,e^{i(p\cdot x-q\cdot y)}\right\} \tag{10}$$

$$=\int\frac{d^3p}{2(2\pi)^3E_{\mathbf{p}}}\left(e^{-ip\cdot(x-y)}-e^{-ip\cdot(y-x)}\right) \tag{11}$$

L&B point out that for spacelike separations of x and y, this commutator is in fact zero. To see this, recall from special relativity that for a spacelike separation, there is always an inertial frame in which the two events are perceived to occur at the same time. Since the integrand in 11 is a Lorentz invariant (see Example 11.2 in L&B for a proof of this), we can evaluate it in any inertial frame, so we can choose the frame where $x^0=y^0$, that is, where the times are equal. In that case, 11 becomes

$$\left[\phi(x),\phi(y)\right]=\int\frac{d^3p}{2(2\pi)^3E_{\mathbf{p}}}\left(e^{-i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}-e^{-i\mathbf{p}\cdot(\mathbf{y}-\mathbf{x})}\right) \tag{12}$$

Since we're integrating **p** over all its values, we can swap $\mathbf{p}\to-\mathbf{p}$ in the second term. This inverts the sign of d^3p, but the integration limits also reverse which introduces another sign change that cancels the first one. Also remember from 2 that $E_{\mathbf{p}}=E_{-\mathbf{p}}$. Thus, we have, for spacelike intervals

$$\left[\phi(x),\phi(y)\right]=\int\frac{d^3p}{2(2\pi)^3E_{\mathbf{p}}}\left(e^{-i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}-e^{-i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}\right)=0 \tag{13}$$

Thus fields separated by a spacelike interval commute, and do not affect each other, so they don't violate special relativity. The same argument does *not* follow for timelike intervals, since for such intervals there is no inertial frame in which the two events have the same times.

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