COMMUTATOR OF REAL SCALAR FIELD WITH MOMENTUM

Link to: physicspages home page.
To leave a comment or report an error, please use the auxiliary blog.
Reference: Tom Lancaster and Stephen J. Blundell, *Quantum Field Theory for the Gifted Amateur*, (Oxford University Press, 2014), Problem 11.2.
Post date: 23 Apr 2019.

The real scalar field $\phi(x)$ can be written as a Fourier expansion

$$\phi(x)=\frac{1}{(2\pi)^{3/2}}\int\frac{d^3p}{\sqrt{2E_p}}\left(a(p)e^{-ip\cdot x}+a^{\dagger}(p)e^{ip\cdot x}\right) \tag{1}$$

where the energy is

$$E_{\mathbf{p}}=+\sqrt{\mathbf{p}^2+m^2} \tag{2}$$

The conjugate momentum density is defined as

$$\Pi^0(x)=\frac{\partial\phi(x)}{\partial t} \tag{3}$$

$$=\frac{i}{(2\pi)^{3/2}}\int\frac{d^3p\;E_p}{\sqrt{2E_p}}\left(-a(p)e^{-ip\cdot x}+a^{\dagger}(p)e^{ip\cdot x}\right) \tag{4}$$

$$=\frac{i}{\sqrt{2}(2\pi)^{3/2}}d^3p\;\sqrt{E_p}\left(a^{\dagger}(p)e^{ip\cdot x}-a(p)e^{-ip\cdot x}\right) \tag{5}$$

The commutators of the creation and annihilation operators are then

$$\left[a(p),a^{\dagger}(q)\right]=\delta^{(3)}(\mathbf{p}-\mathbf{q}) \tag{6}$$

$$[a(p),a(q)]=\left[a^{\dagger}(p),a^{\dagger}(q)\right]=0 \tag{7}$$

From these we can work out the following commutator at equal times, that is, when $x^0=y^0$:

$$\left[\phi\left(x\right),\Pi^{0}\left(y\right)\right]=\frac{i}{2\left(2\pi\right)^{3}}\int\frac{d^{3}p\,d^{3}q}{\sqrt{E_{p}}}\sqrt{E_{q}}\left[a\left(p\right)e^{-i\mathbf{p}\cdot\mathbf{x}}+a^{\dagger}\left(p\right)e^{i\mathbf{p}\cdot\mathbf{x}},a^{\dagger}\left(q\right)e^{i\mathbf{q}\cdot\mathbf{y}}-a\left(q\right)e^{-i\mathbf{q}\cdot\mathbf{y}}\right] \quad (8)$$

$$=\frac{i}{2\left(2\pi\right)^{3}}\int\frac{d^{3}p\,d^{3}q}{\sqrt{E_{p}}}\sqrt{E_{q}}\left\{ \delta^{(3)}\left(\mathbf{p}-\mathbf{q}\right)e^{-i\left(\mathbf{p}\cdot\mathbf{x}-\mathbf{q}\cdot\mathbf{y}\right)}+\delta^{(3)}\left(\mathbf{p}-\mathbf{q}\right)e^{i\left(\mathbf{p}\cdot\mathbf{x}-\mathbf{q}\cdot\mathbf{y}\right)}\right\} \quad (9)$$

$$=\frac{i}{2\left(2\pi\right)^{3}}\int d^{3}p\left(e^{-i\mathbf{p}\cdot\left(\mathbf{x}-\mathbf{y}\right)}+e^{i\mathbf{p}\cdot\left(\mathbf{x}-\mathbf{y}\right)}\right) \quad (10)$$

Because we're integrating over all values of $\mathbf{p}$, we can swap $\mathbf{p}\rightarrow-\mathbf{p}$ in the second term. This introduces a sign change in $d^{3}p$ and another sign change from the reversal of the integration limits, so the two signs cancel each other, giving

$$\left[\phi\left(x\right),\Pi^{0}\left(y\right)\right]=\frac{2i}{2\left(2\pi\right)^{3}}\int d^{3}p\,e^{-i\mathbf{p}\cdot\left(\mathbf{x}-\mathbf{y}\right)} \quad (11)$$

$$=i\delta^{(3)}\left(\mathbf{x}-\mathbf{y}\right) \quad (12)$$

PINGBACKS

Pingback: Commutator of conserved charge with field