COMMUTATORS OF COMPLEX SCALAR FIELDS AT GENERAL TIMES

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Reference: Tom Lancaster and Stephen J. Blundell, *Quantum Field Theory for the Gifted Amateur*, (Oxford University Press, 2014), Problem 12.2.

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The complex scalar field $\psi(x)$ has the mode expansion

The notation differs from Lahiri & Pal in that $\hat{a}$ becomes b and ϕ becomes ψ.

$$\psi(x)=\int\frac{d^3p}{\sqrt{(2\pi)^3 2E_p}}\left(a(p)e^{-ip\cdot x}+b^{\dagger}(p)e^{ip\cdot x}\right) \tag{1}$$

$$\psi^{\dagger}(x)=\int\frac{d^3p}{\sqrt{(2\pi)^3 2E_p}}\left(b(p)e^{-ip\cdot x}+a^{\dagger}(p)e^{ip\cdot x}\right) \tag{2}$$

We can work out the commutator of the field at different spacetime points using the commutators for the operators

$$\left[a(p),a^{\dagger}(q)\right]=\left[b(p),b^{\dagger}(q)\right]=\delta^{(3)}(\mathbf{p}-\mathbf{q}) \tag{3}$$

with all other commutators being zero. To simplify the notation, I'll use

$$a_{\mathbf{p}}\equiv a(p) \tag{4}$$

and similar for other operators.

We get

$$\left[\psi(x),\psi^{\dagger}(y)\right]=\frac{1}{2(2\pi)^3}\int\frac{d^3p\,d^3q}{\sqrt{E_{\mathbf{p}}E_{\mathbf{q}}}}\left\{\left[a_{\mathbf{p}},a_{\mathbf{q}}^{\dagger}\right]e^{-i(p\cdot x-q\cdot y)}-\left[b_{\mathbf{q}},b_{\mathbf{p}}^{\dagger}\right]e^{i(p\cdot x-q\cdot y)}\right\} \tag{5}$$

$$=\frac{1}{2(2\pi)^3}\int\frac{d^3p\,d^3q}{\sqrt{E_{\mathbf{p}}E_{\mathbf{q}}}}\left\{\delta^{(3)}(\mathbf{p}-\mathbf{q})e^{-i(p\cdot x-q\cdot y)}-\delta^{(3)}(\mathbf{p}-\mathbf{q})e^{i(p\cdot x-q\cdot y)}\right\} \tag{6}$$

$$=\frac{1}{2(2\pi)^3}\int\frac{d^3p}{E_{\mathbf{p}}}\left\{e^{-ip\cdot(x-y)}-e^{ip\cdot(x-y)}\right\} \tag{7}$$

Note that if $x^0=y^0$ (that is, the times are equal), then

$$x-y=[0,\mathbf{x}-\mathbf{y}] \tag{8}$$

and we get

$$\left[\psi(t,\mathbf{x}),\psi^{\dagger}(t,\mathbf{y})\right]=\frac{1}{2(2\pi)^{3}}\int\frac{d^{3}p}{E_{\mathbf{p}}}\left\{e^{-i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}-e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}\right\} \tag{9}$$

Since we're integrating over all $\mathbf{p}$, we can swap $\mathbf{p}\to-\mathbf{p}$ in the second term with the result

$$\left[\psi(t,\mathbf{x}),\psi^{\dagger}(t,\mathbf{y})\right]=0 \tag{10}$$

This is the complex scalar field analog to the result we got earlier for a real scalar field, which shows that for two spacetime points with a spacelike separation, the commutator of the fields at these two points is zero, which is required to preserve causality.

In their section 12.3, L&B consider the non-relativistic limit of a complex scalar field. They define the field as

$$\phi(\mathbf{x},t)=\Psi(\mathbf{x},t)\,e^{-imc^{2}t/\hbar} \tag{11}$$

where the rest energy has been factored out, since in the non-relativistic case, the rest energy is by far the dominant part of the energy. They then show that we can write the other part of the field as

$$\Psi(x)=\int\frac{d^{3}p}{(2\pi)^{3/2}}a_{\mathbf{p}}e^{-ip\cdot x} \tag{12}$$

Using this form, we have for the commutator at general spacetime points:

$$\left[\Psi(x),\Psi^{\dagger}(y)\right]=\frac{1}{(2\pi)^{3}}\int d^{3}p\,d^{3}q\,\left[a_{\mathbf{p}},a_{\mathbf{q}}^{\dagger}\right]e^{-i(p\cdot x-q\cdot y)} \tag{13}$$

$$=\frac{1}{(2\pi)^{3}}\int d^{3}p\,d^{3}q\,\delta^{(3)}(\mathbf{p}-\mathbf{q})\,e^{-i(p\cdot x-q\cdot y)} \tag{14}$$

$$=\frac{1}{(2\pi)^{3}}\int d^{3}p\,e^{-ip\cdot(x-y)} \tag{15}$$

At equal times, we have

$$\left[\Psi(t,\mathbf{x}),\Psi^{\dagger}(t,\mathbf{y})\right]=\frac{1}{(2\pi)^{3}}\int d^{3}p\,e^{-i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} \tag{16}$$

$$=\delta^{(3)}(\mathbf{x}-\mathbf{y}) \tag{17}$$

which agrees with L&B's equation 12.27.

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