COMMUTATORS OF COMPLEX SCALAR FIELD WITH CONSERVED CHARGE

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Reference: Tom Lancaster and Stephen J. Blundell, *Quantum Field Theory for the Gifted Amateur*, (Oxford University Press, 2014), Problem 12.3.

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Given two scalar fields ϕ_1 and ϕ_2, we can form a complex scalar field and its conjugate by the following.

$$\psi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2) \tag{1}$$

$$\psi^\dagger = \frac{1}{\sqrt{2}}(\phi_1 - i\phi_2) \tag{2}$$

The relations can be inverted to give

$$\phi_1 = \frac{1}{\sqrt{2}}\left(\psi + \psi^\dagger\right) \tag{3}$$

$$\phi_2 = -\frac{i}{\sqrt{2}}\left(\psi - \psi^\dagger\right) \tag{4}$$

If the complex scalar field has an internal symmetry, then the transformation

$$\psi \to e^{i\alpha}\psi \tag{5}$$

$$\psi^\dagger \to e^{-i\alpha}\psi^\dagger \tag{6}$$

leave the Lagrangian unchanged. For an infinitesimal transformation, with $\delta\alpha$ very small, we have

$$\psi \to \psi + i\psi\delta\alpha \tag{7}$$

$$\psi^\dagger \to \psi^\dagger - i\psi^\dagger\delta\alpha \tag{8}$$

From this, we can get the quantities

$$D\psi = \left.\frac{\partial\psi}{\partial\alpha}\right|_{\alpha=0} = i\psi \tag{9}$$

$$D\psi^\dagger = \left.\frac{\partial\psi^\dagger}{\partial\alpha}\right|_{\alpha=0} = -i\psi^\dagger \tag{10}$$

Using the relation for the commutator of the conserved Noether charge

$$[Q_N, \phi_i] = -iD\phi_i \tag{11}$$

we have, using 3

$$[Q_N, \phi_1] = \frac{1}{\sqrt{2}}\left[Q_N, \psi + \psi^\dagger\right] \tag{12}$$

$$= \frac{-i}{\sqrt{2}}\left(D\psi + D\psi^\dagger\right) \tag{13}$$

$$= \frac{-i}{\sqrt{2}}\left(i\psi - i\psi^\dagger\right) \tag{14}$$

$$= \frac{1}{\sqrt{2}}\left(\psi - \psi^\dagger\right) \tag{15}$$

$$= i\phi_2 \tag{16}$$

Similarly, we get

$$[Q_N, \phi_2] = -\frac{i}{\sqrt{2}}\left[Q_N, \psi - \psi^\dagger\right] \tag{17}$$

$$= -\frac{1}{\sqrt{2}}\left(D\psi - D\psi^\dagger\right) \tag{18}$$

$$= -\frac{1}{\sqrt{2}}\left(i\psi + i\psi^\dagger\right) \tag{19}$$

$$= -\frac{i}{\sqrt{2}}\left(\psi + \psi^\dagger\right) \tag{20}$$

$$= -i\phi_1 \tag{21}$$

Combining these results, we have

$$[Q_N, \psi] = \frac{1}{\sqrt{2}}[Q_N, \phi_1 + i\phi_2] \tag{22}$$

$$= \frac{1}{\sqrt{2}}(i\phi_2 + \phi_1) \tag{23}$$

$$= \psi \tag{24}$$