NONRELATIVISTIC COMPLEX SCALAR FIELD: COMMUTATOR OF NUMBER WITH PHASE

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog.

Reference: Tom Lancaster and Stephen J. Blundell, *Quantum Field Theory for the Gifted Amateur*, (Oxford University Press, 2014), Problem 12.4.

Post date: 7 May 2019.

In their section 12.3, L&B write the nonrelativistic version of the complex scalar field in the form

$$\Psi(x)=\sqrt{\rho(x)}e^{i\theta(x)} \tag{1}$$

so that $\rho(x)=|\Psi(x)|^2$ is the probability density and $\theta(x)$ is a spacetime-dependent phase. Both $\rho$and θ are taken to be real functions.

Using this form, the nonrelatistic Lagrangian becomes

$$\mathcal{L}=\frac{i}{2}\partial_0\rho-\rho\partial_0\theta-\frac{1}{2m}\left[\frac{1}{4\rho}(\nabla\rho)^2+\rho(\nabla\theta)^2\right]-\frac{g}{2}\rho^2 \tag{2}$$

where g represents the interaction term.

If we make the global transformation $\theta\to\theta+\alpha$ where α is a constant, the Lagrangian is unchanged, since only the derivatives of θ enter into the Lagrangian.

To apply Noether's theorem to this transformation, we have

$$D\theta=\left.\frac{\partial\theta'}{\partial\alpha}\right|_{\alpha=0} \tag{3}$$

$$=\left.\frac{\partial(\theta+\alpha)}{\partial\alpha}\right|_{\alpha=0} \tag{4}$$

$$=1 \tag{5}$$

Since we're not transforming ρ, $D\rho=0$. The zero component of the Noether current is then

$$J_N^0 = \Pi_\theta^0 D\theta + \Pi_\rho^0 D\rho \tag{6}$$
$$= \frac{\partial \mathcal{L}}{\partial (\partial_0 \theta)} D\theta + 0 \tag{7}$$
$$= -\rho(x) \times 1 \tag{8}$$
$$= -\rho(x) \tag{9}$$

which gives a conserved charge of

$$Q_N = \int d^3x \; J_N^0(x) \tag{10}$$
$$= -\int d^3x \; \rho(x) \tag{11}$$

This gives what L&B call the 'conventional' charge

$$Q_{Nc} = - :Q_N: \tag{12}$$
$$=: \int d^3x \; \rho(x): \tag{13}$$

I'm using the usual double-colon notation for normal ordering rather than L&B's N symbol, since there are too many Ns.

We can now use the commutator of conserved charge together with 5 to write

$$[Q_N, \theta] = -iD\theta = -i \tag{14}$$

If we define

$$N(t) = \int d^3x \; \rho(x) \tag{15}$$

to be the total number of particles, then

$$N(t) = -Q_N \tag{16}$$

and

$$[Q_N, \theta] = -[N(t), \theta(\mathbf{x}, t)] = -i \tag{17}$$

or

$$[N(t), \theta(\mathbf{x}, t)] = i \tag{18}$$