COMPLEX SCALAR FIELD: INTERNAL TRANSFORMATION

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Reference: Tom Lancaster and Stephen J. Blundell, *Quantum Field Theory for the Gifted Amateur*, (Oxford University Press, 2014), Problem 12.7.

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The complex scalar field considered by L&B in their Chapter 12 is invariant under the internal transformation

$$\psi \to \psi' = e^{i\alpha}\psi \tag{1}$$

They point out that this transformation can also be written using the unitary operator

$$U(\alpha) = e^{iQ_{Nc}\alpha} \tag{2}$$

where Q_{Nc} is the conserved number charge operator, which is defined in terms of the Noether charge Q_N by normal ordering:

$$Q_{Nc} = -:Q_N: \tag{3}$$

To verify that this unitary operator works correctly, we consider $U^\dagger\psi U$ for an infinitesimal α. We have, keeping only terms up to first order in α

$$U^\dagger(\alpha)\psi U(\alpha) = (1 - iQ_{Nc}\alpha)\psi(1 + iQ_{Nc}\alpha) \tag{4}$$

$$= \psi - i\alpha[Q_{Nc}, \psi] \tag{5}$$

We can now use the formula

$$[Q_N, \psi] = -iD\psi \tag{6}$$

where

$$D\psi = \left.\frac{\partial\psi'}{\partial\alpha}\right|_{\alpha=0} = i\psi \tag{7}$$

From 3 we have

$$[Q_{Nc}, \psi] = -[Q_N, \psi] = -\psi \tag{8}$$

so we have

$$U^\dagger(\alpha)\psi U(\alpha) = \psi(1 + i\alpha) \tag{9}$$

Taking α to be finite, we can apply this formula n times and let $n \to \infty$ to get

$$U^{\dagger}(\alpha)\psi U(\alpha) = \lim_{n\to\infty}\psi\left(1+i\frac{\alpha}{n}\right)^{n} \tag{10}$$

$$= e^{i\alpha}\psi \tag{11}$$

Thus the unitary transformation is equivalent to the transformation 1.

In case you're worried that the normal ordering done to get from Q_N to Q_{Nc} messes up the relation 8, we can see that this isn't a problem by considering Q_{Nc} and ψ in their mode expansions. From L&B's equations 12.5 and 12.15, we have

$$\psi(x) = \int \frac{d^3p}{(2\pi)^{3/2}\sqrt{2E_{\mathbf{p}}}}\left(a_{\mathbf{p}}e^{-ip\cdot x} + b_{\mathbf{p}}^{\dagger}e^{ip\cdot x}\right) \tag{12}$$

$$Q_{Nc} = \int d^3q\left(n_{\mathbf{q}}^{(a)} - n_{\mathbf{q}}^{(b)}\right) \tag{13}$$

$$= \int d^3q\left(a_{\mathbf{q}}^{\dagger}a_{\mathbf{q}} - b_{\mathbf{q}}^{\dagger}b_{\mathbf{q}}\right) \tag{14}$$

Taking the commutator explicitly and using

$$\left[a_{\mathbf{p}}, a_{\mathbf{q}}^{\dagger}\right] = \left[b_{\mathbf{p}}, b_{\mathbf{q}}^{\dagger}\right] = \delta(\mathbf{p}-\mathbf{q}) \tag{15}$$

with all other commutators being zero, we have

$$[Q_{Nc},\psi] = \int \frac{d^3p\, d^3q}{(2\pi)^{3/2}\sqrt{2E_{\mathbf{p}}}}\left\{\left[a_{\mathbf{q}}^{\dagger}, a_{\mathbf{p}}\right]a_{\mathbf{q}}e^{-ip\cdot x} - b_{\mathbf{q}}^{\dagger}\left[b_{\mathbf{q}}, b_{\mathbf{p}}^{\dagger}\right]e^{ip\cdot x}\right\} \tag{16}$$

$$= \int \frac{d^3p\, d^3q}{(2\pi)^{3/2}\sqrt{2E_{\mathbf{p}}}}\left\{-\left[a_{\mathbf{p}}, a_{\mathbf{q}}^{\dagger}\right]a_{\mathbf{q}}e^{-ip\cdot x} - b_{\mathbf{q}}^{\dagger}\left[b_{\mathbf{q}}, b_{\mathbf{p}}^{\dagger}\right]e^{ip\cdot x}\right\} \tag{17}$$

$$= \int \frac{d^3p\, d^3q}{(2\pi)^{3/2}\sqrt{2E_{\mathbf{p}}}}\left\{-\delta(\mathbf{p}-\mathbf{q})\,a_{\mathbf{q}}e^{-ip\cdot x} - b_{\mathbf{q}}^{\dagger}\delta(\mathbf{p}-\mathbf{q})\,e^{ip\cdot x}\right\} \tag{18}$$

$$= -\int \frac{d^3p}{(2\pi)^{3/2}\sqrt{2E_{\mathbf{p}}}}\left(a_{\mathbf{p}}e^{-ip\cdot x} + b_{\mathbf{p}}^{\dagger}e^{ip\cdot x}\right) \tag{19}$$

$$= -\psi \tag{20}$$