THREE-COMPONENT FIELD: NUMBER OPERATOR

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog.

Reference: Tom Lancaster and Stephen J. Blundell, *Quantum Field Theory for the Gifted Amateur*, (Oxford University Press, 2014), Problem 13.1.

Post date: 7 May 2019.

In their Chapter 13, L&B consider a field $\mathbf{\Phi}$ which has 3 components that could represent 3 different states of a particle. They give the Lagrangian as

$$\mathcal{L}=\frac{1}{2}\left(\partial^{\mu}\mathbf{\Phi}\right)\cdot\left(\partial_{\mu}\mathbf{\Phi}\right)-\frac{m^{2}}{2}\mathbf{\Phi}\cdot\mathbf{\Phi} \tag{1}$$

This Lagrangian is invariant under 3-d rotations which they show leads to a conserved charge vector $\boldsymbol{Q}_{Nc}$:

$$\boldsymbol{Q}_{Nc}=\int d^{3}x\,\left(\mathbf{\Phi}\times\partial_{0}\mathbf{\Phi}\right) \tag{2}$$

where the integrand is the standard cross product of two 3-d vectors.

The field vector $\mathbf{\Phi}$ is assumed to have the mode expansion for each component given by

$$\Phi_{\alpha}=\int\frac{d^{3}p}{(2\pi)^{3/2}\sqrt{2E_{\mathbf{p}}}}\left(a_{\mathbf{p}\alpha}e^{-ip\cdot x}+a_{\mathbf{p}\alpha}^{\dagger}e^{ip\cdot x}\right) \tag{3}$$

where the index $\alpha=1,2,3$ specifies which of the 3 states of the particle we're looking at.

With this mode decomposition, the conserved charge then has components

$$Q_{Nc}^{a}=-i\int d^{3}p\,\varepsilon^{abc}a_{\mathbf{p}b}^{\dagger}a_{\mathbf{p}c} \tag{4}$$

For example

$$Q_{Nc}^{3}=-i\int d^{3}p\left(a_{\mathbf{p}1}^{\dagger}a_{\mathbf{p}2}-a_{\mathbf{p}2}^{\dagger}a_{\mathbf{p}1}\right) \tag{5}$$

The first part of the problem is to write the conserved charge in the compact form

$$\boldsymbol{Q}_{Nc}=\int d^{3}p\,\boldsymbol{A}_{\mathbf{p}}^{\dagger}\boldsymbol{J}\boldsymbol{A}_{\mathbf{p}} \tag{6}$$

where $\boldsymbol{A}=\left(a_{\mathbf{p}1},a_{\mathbf{p}2},a_{\mathbf{p}3}\right)$ is the vector of annihilation operators and $\boldsymbol{J}$ is a 3-component vector where each component is a 3×3 matrix consisting of the spin-1 angular momentum matrices given in L&B's Chapter 9. Actually, the matrices they give there are 4×4, so I'm assuming they want us to use the 3×3 sub-matrix containing the spatial components. These matrices are

The index on each J_i has been lowered from that given in Chapter 9, so the sign is reversed.

$$J_x=-i\begin{bmatrix}0&0&0\\0&0&-1\\0&1&0\end{bmatrix} \tag{7}$$

$$J_y=-i\begin{bmatrix}0&0&1\\0&0&0\\-1&0&0\end{bmatrix} \tag{8}$$

$$J_z=-i\begin{bmatrix}0&-1&0\\1&0&0\\0&0&0\end{bmatrix} \tag{9}$$

We can now work out the matrix term $\boldsymbol{A}^{\dagger}\boldsymbol{J}\boldsymbol{A}$. (I'll drop the $\mathbf{p}$ subscript to avoid clutter.) First, we have

$$\boldsymbol{J}\boldsymbol{A}=-i\begin{bmatrix}0&0&0\\0&0&-a_1\\0&a_1&0\end{bmatrix}-i\begin{bmatrix}0&0&a_2\\0&0&0\\-a_2&0&0\end{bmatrix}-i\begin{bmatrix}0&-a_3&0\\a_3&0&0\\0&0&0\end{bmatrix} \tag{10}$$

$$=-i\begin{bmatrix}0&-a_3&a_2\\a_3&0&-a_1\\-a_2&a_1&0\end{bmatrix} \tag{11}$$

Next, we have

$$\boldsymbol{A}^{\dagger}\boldsymbol{J}\boldsymbol{A}=-i\left[\begin{array}{ccc}a_1^{\dagger}&a_2^{\dagger}&a_3^{\dagger}\end{array}\right]\begin{bmatrix}0&-a_3&a_2\\a_3&0&-a_1\\-a_2&a_1&0\end{bmatrix} \tag{12}$$

$$=-i\left[\begin{array}{ccc}a_2^{\dagger}a_3-a_3^{\dagger}a_2&a_3^{\dagger}a_1-a_1^{\dagger}a_3&a_1^{\dagger}a_2-a_2^{\dagger}a_1\end{array}\right] \tag{13}$$

By comparing this with 4, we see that the 3 components of $\boldsymbol{Q}_{Nc}$ match up properly.

Next, we want to change to a different set of creation and annihilation operators, namely those given in Exercise 3.3

$$\hat{b}_1^\dagger \equiv -\frac{1}{\sqrt{2}}\left(\hat{a}_1^\dagger + i\hat{a}_2^\dagger\right) \tag{14}$$

$$\hat{b}_0^\dagger \equiv \hat{a}_3^\dagger \tag{15}$$

$$\hat{b}_{-1}^\dagger \equiv \frac{1}{\sqrt{2}}\left(\hat{a}_1^\dagger - i\hat{a}_2^\dagger\right) \tag{16}$$

$$\hat{b}_1 \equiv -\frac{1}{\sqrt{2}}\left(\hat{a}_1 - i\hat{a}_2\right) \tag{17}$$

$$\hat{b}_0 \equiv \hat{a}_3 \tag{18}$$

$$\hat{b}_{-1} \equiv \frac{1}{\sqrt{2}}\left(\hat{a}_1 + i\hat{a}_2\right) \tag{19}$$

Using $\boldsymbol{B} = (b_1, b_0, b_{-1})$ we are to find the new set of matrices $\boldsymbol{J}$ so that

$$\boldsymbol{Q}_{Nc} = \int d^3p\; \boldsymbol{B}_{\mathbf{p}}^\dagger \boldsymbol{J} \boldsymbol{B}_{\mathbf{p}} \tag{20}$$

There may be some quick way of doing this, but it seems that we need to express the a_i operators in terms of b_j and proceed from there. We have

$$a_1 = \frac{1}{\sqrt{2}}\left(b_{-1} - b_1\right) \tag{21}$$

$$a_2 = -\frac{i}{\sqrt{2}}\left(b_{-1} + b_1\right) \tag{22}$$

$$a_3 = b_0 \tag{23}$$

$$a_1^\dagger = \frac{1}{\sqrt{2}}\left(b_{-1}^\dagger - b_1^\dagger\right) \tag{24}$$

$$a_2^\dagger = \frac{i}{\sqrt{2}}\left(b_{-1}^\dagger + b_1^\dagger\right) \tag{25}$$

$$a_3^\dagger = b_0^\dagger \tag{26}$$

With these terms, we have

$$a_2^\dagger a_3 - a_3^\dagger a_2 = -\frac{i}{\sqrt{2}}\left(b_1^\dagger b_0 - b_0^\dagger\left(b_1 + b_{-1}\right) + b_{-1}^\dagger b_0\right) \tag{27}$$

$$a_3^\dagger a_1 - a_1^\dagger a_3 = \frac{1}{\sqrt{2}}\left(b_1^\dagger b_0 - b_0^\dagger\left(b_1 - b_{-1}\right) - b_{-1}^\dagger b_0\right) \tag{28}$$

$$a_1^\dagger a_2 - a_2^\dagger a_1 = i\left(b_1^\dagger b_1 - b_{-1}^\dagger b_{-1}\right) \tag{29}$$

Pulling out the creation operators we have

$$\boldsymbol{B}_{\mathbf{p}}^{\dagger}\boldsymbol{J}\boldsymbol{B}_{\mathbf{p}}=-i\left[\begin{array}{ccc} b_1^{\dagger} & b_0^{\dagger} & b_{-1}^{\dagger}\end{array}\right]\left[\begin{array}{ccc} -\frac{i}{\sqrt{2}}b_0 & \frac{1}{\sqrt{2}}b_0 & ib_1\\ \frac{i}{\sqrt{2}}\left(b_1+b_{-1}\right) & -\frac{1}{\sqrt{2}}\left(b_1-b_{-1}\right) & 0\\ -\frac{i}{\sqrt{2}}b_0 & -\frac{1}{\sqrt{2}}b_0 & -ib_{-1}\end{array}\right] \tag{30}$$

We now need to express the 3×3 matrix on the RHS as a product of 3 separate matrices multiplied into $\boldsymbol{B}=(b_1,b_0,b_{-1})$. We therefore have

$$J_1=-i\left[\begin{array}{ccc} 0 & 0 & i\\ \frac{i}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0\\ 0 & 0 & 0\end{array}\right]=\left[\begin{array}{ccc} 0 & 0 & 1\\ \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0\\ 0 & 0 & 0\end{array}\right] \tag{31}$$

$$J_2=-i\left[\begin{array}{ccc} -\frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0\\ 0 & 0 & 0\\ -\frac{i}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0\end{array}\right]=\left[\begin{array}{ccc} -\frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} & 0\\ 0 & 0 & 0\\ -\frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0\end{array}\right] \tag{32}$$

$$J_3=-i\left[\begin{array}{ccc} 0 & 0 & 0\\ \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0\\ 0 & 0 & -i\end{array}\right]=\left[\begin{array}{ccc} 0 & 0 & 0\\ \frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} & 0\\ 0 & 0 & -1\end{array}\right] \tag{33}$$

PINGBACKS

Pingback: Massive electromagnetism: Polarization vectors