MASSIVE ELECTROMAGNETISM: POLARIZATION VECTORS

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Reference: Tom Lancaster and Stephen J. Blundell, *Quantum Field Theory for the Gifted Amateur*, (Oxford University Press, 2014), Problem 13.2.

Post date: 13 May 2019.

The Lagrangian for the massive electromagnetic field is

$$\mathcal{L}=-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}+\frac{1}{2}m^{2}A_{\mu}A^{\mu} \tag{1}$$

The equations of motion, derived by applying the Euler-Lagrange equations to 1 are given by L&B's equation 13.18:

$$\partial_{\mu}F^{\mu\nu}+m^{2}A^{\nu}=0 \tag{2}$$

Using the equation

$$\partial_{\mu}\partial_{\nu}F^{\mu\nu}=\partial_{\mu}\partial_{\nu}\left(\partial^{\mu}A^{\nu}-\partial^{\nu}A^{\mu}\right)=0 \tag{3}$$

we see that this imposes the condition

$$\partial_{\mu}A^{\mu}=0 \tag{4}$$

If the field can be expanded in a Fourier transform with terms like $a_{\mathbf{p}}e^{-ip\cdot x}$ then this condition implies that $p_{\mu}A^{\mu}=0$, which in turn implies that the four components of A^{μ} are not independent.

With these constraints, we can now consider the mode expansion of the field A^{μ}. Unlike the three-component field $\mathbf{\Phi}$ we considered earlier, the field A^{μ} is a genuine four-vector, so the Fourier expansion must also be a four-vector. To this end, we introduce a set of three polarization vectors $\epsilon_{\lambda}^{\mu}(p)$. Here, the index $\lambda=1,2,3$ labels which vector we're considering, and the index $\mu=0,1,2,3$ labels the component of the vector. These polarization vectors are similar to those we've seen for 'proper' electromagnetism, but in that case, we couldn't impose the condition 4, so we had to use four vectors rather than three. The expansion is

$$A^{\mu}(x)=\int\frac{d^{3}p}{\sqrt{(2\pi)^{3}2E_{\mathbf{p}}}}\sum_{\lambda=1}^{3}\left[\epsilon_{\lambda}^{\mu}(p)\,a_{\lambda\mathbf{p}}e^{-ip\cdot x}+\epsilon_{\lambda}^{\mu*}(p)\,a_{\lambda\mathbf{p}}^{\dagger}e^{ip\cdot x}\right] \tag{5}$$

The condition 4 implies that

$$p_\mu \epsilon_\lambda^\mu (p) = 0 \tag{6}$$

for all λ. In addition, we impose the orthonormality condition

$$\epsilon_\lambda^* (p) \cdot \epsilon_{\lambda'} (p) = g_{\mu\nu} \epsilon_\lambda^{\mu *} \epsilon_{\lambda'}^\nu = -\delta_{\lambda\lambda'} \tag{7}$$

Beyond this, we have a number of choices for the ϵ_λs. If we choose a particle at rest with linear polarization, then we can have

$$\epsilon_1 (m, 0) = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \tag{8}$$

$$\epsilon_2 (m, 0) = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \tag{9}$$

$$\epsilon_3 (m, 0) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \tag{10}$$

To get the polarization vectors when the particle has a momentum $p^\mu = (E_{\mathbf{p}}, 0, 0, |\mathbf{p}|)$ (it's moving in the z direction), we can apply a Lorentz transformation of the form

$$\Lambda^\mu_{\ \nu} (p) = \begin{bmatrix} \gamma & 0 & 0 & \gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \gamma\beta & 0 & 0 & \gamma \end{bmatrix} \tag{11}$$

where as usual β is the speed in units with $c = 1$ and

$$\gamma = \frac{1}{\sqrt{1 - \beta^2}} \tag{12}$$

The energy and momentum are given by $E_{\mathbf{p}} = \gamma m$ and $|\mathbf{p}| = \gamma m \beta$, so is equivalent to

$$\Lambda^\mu_{\ \nu} (p) = \frac{1}{m} \begin{bmatrix} E_{\mathbf{p}} & 0 & 0 & |\mathbf{p}| \\ 0 & m & 0 & 0 \\ 0 & 0 & m & 0 \\ |\mathbf{p}| & 0 & 0 & E_{\mathbf{p}} \end{bmatrix} \tag{13}$$

Applying this transformation to the three ϵ_λs in 9 we get

$$\epsilon_1(E_{\mathbf{p}},0,0,|\mathbf{p}|) = \Lambda^{\mu}_{\ \nu}(p)\,\epsilon_1(m,0) = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \tag{14}$$

$$\epsilon_2(E_{\mathbf{p}},0,0,|\mathbf{p}|) = \Lambda^{\mu}_{\ \nu}(p)\,\epsilon_2(m,0) = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \tag{15}$$

$$\epsilon_3(E_{\mathbf{p}},0,0,|\mathbf{p}|) = \Lambda^{\mu}_{\ \nu}(p)\,\epsilon_3(m,0) = \begin{bmatrix} |\mathbf{p}|/m \\ 0 \\ 0 \\ E_{\mathbf{p}}/m \end{bmatrix} \tag{16}$$

These vectors are still correctly normalized, as we can see from direct calculation. The normalizations of ϵ_1 and ϵ_2 are fairly obvious. For ϵ_3 we have

$$\begin{aligned}
\epsilon_3^* \cdot \epsilon_3 &= g_{\mu\nu}\epsilon_3^{\mu *}\epsilon_3^{\nu} && (17)\\
&= \frac{1}{m^2}\left(\mathbf{p}^2 - E_{\mathbf{p}}^2\right) && (18)\\
&= \frac{\mathbf{p}^2 - \mathbf{p}^2 - m^2}{m^2} && (19)\\
&= -1 && (20)
\end{aligned}$$

which agrees with 7.

If we use circular polarization, we can have

$$\epsilon_R^* = -\frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ i \\ 0 \end{bmatrix} \tag{21}$$

$$\epsilon_L^* = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ -i \\ 0 \end{bmatrix} \tag{22}$$

$$\epsilon_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \tag{23}$$

where the subscripts R and L stand for right and left. These are also normalized, as we can see:

$$\epsilon_R^* \cdot \epsilon_R = g_{\mu\nu} \epsilon_R^{\mu *} \epsilon_R^{\nu} \tag{24}$$

$$= \frac{1}{2}(-1-1) \tag{25}$$

$$= -1 \tag{26}$$

$$\epsilon_L^* \cdot \epsilon_L = g_{\mu\nu} \epsilon_L^{\mu *} \epsilon_L^{\nu} \tag{27}$$

$$= \frac{1}{2}(-1-1) \tag{28}$$

$$= -1 \tag{29}$$

$$\epsilon_R^* \cdot \epsilon_L = g_{\mu\nu} \epsilon_R^{\mu *} \epsilon_L^{\nu} \tag{30}$$

$$= -\frac{1}{2}(-1+1) \tag{31}$$

$$= 0 \tag{32}$$

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