MASSIVE ELECTROMAGNETISM: PROJECTION OPERATORS

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Reference: Tom Lancaster and Stephen J. Blundell, *Quantum Field Theory for the Gifted Amateur*, (Oxford University Press, 2014), Problem 13.3.

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In their section 13.3, L&B show that the projection of a four-vector along a given four-momentum p^{μ} is given by the longitudinal projection operator

$$P_L^{\mu\nu} = \frac{p^{\mu}p^{\nu}}{p^2} \tag{1}$$

The projection transverse to the momentum is then given by

$$P_T^{\mu\nu} = g^{\mu\nu} - \frac{p^{\mu}p^{\nu}}{p^2} \tag{2}$$

For the particular case of massive electromagnetism, they show that the projection operator can be written in terms of the polarization vectors ϵ_{λ}:

$$\sum_{\lambda=1}^{3} \epsilon_{\lambda\mu}(p)\,\epsilon_{\lambda\nu}(p) = -P_{\mu\nu}^{T} \tag{3}$$

One property satisfied by a projection operator is that applying it twice (or any number of times, actually) gives the same result as applying it just once. In other words, once you've projected a given vector X^{μ} onto a given direction, projecting the result again won't change anything. We can verify this by showing that $P_L^2 = P_L$ and $P_T^2 = P_T$. For this purpose, consider the action on the given vector X^{μ}.

$$X_L^{\mu} = P_L^{\mu\nu} X_{\nu} = \frac{p^{\mu}p^{\nu}}{p^2} X_{\nu} \tag{4}$$

Now apply P_L again:

$$P_L^{\alpha\mu} X_{L\mu} = \frac{p^{\alpha}p^{\mu}}{p^2}\frac{p_{\mu}p^{\nu}}{p^2} X_{\nu} \tag{5}$$

$$= \frac{p^{\alpha}p^{\nu}p^2}{p^4} X_{\nu} \tag{6}$$

$$= \frac{p^{\alpha}p^{\nu}}{p^2} X_{\nu} \tag{7}$$

Thus $P_L^2 = P_L$.

For P_T we have

$$X_T^{\mu} = P_T^{\mu\nu} X_{\nu} \tag{8}$$

$$= \left(g^{\mu\nu} - \frac{p^{\mu}p^{\nu}}{p^2}\right) X_{\nu} \tag{9}$$

$$= X^{\mu} - \frac{p^{\mu}p^{\nu}}{p^2} X_{\nu} \tag{10}$$

Now apply P_T again:

$$P_T^{\alpha\mu} X_{T\mu} = \left(g^{\alpha\mu} - \frac{p^{\alpha}p^{\mu}}{p^2}\right)\left(X_{\mu} - \frac{p_{\mu}p^{\nu}}{p^2} X_{\nu}\right) \tag{11}$$

$$= X^{\alpha} - \frac{p^{\alpha}p^{\mu}}{p^2} X_{\mu} - g^{\alpha\mu}\frac{p_{\mu}p^{\nu}}{p^2} X_{\nu} + \frac{p^{\alpha}p^{\mu}}{p^2}\frac{p_{\mu}p^{\nu}}{p^2} X_{\nu} \tag{12}$$

$$= X^{\alpha} - 2\frac{p^{\alpha}p^{\mu}}{p^2} X_{\mu} + \frac{p^{\alpha}p^{\mu}}{p^2} X_{\mu} \tag{13}$$

$$= X^{\alpha} - \frac{p^{\alpha}p^{\mu}}{p^2} X_{\mu} \tag{14}$$

$$= \left(g^{\alpha\mu} - \frac{p^{\alpha}p^{\mu}}{p^2}\right) X_{\mu} \tag{15}$$

In the third line, after summing over the index μ, we've relabelled ν as μ, which is always permissible for a summed index. Thus $P_T^2 = P_T$.

PINGBACKS

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