ELECTROMAGNETISM: LAGRANGIAN USING PROJECTION OPERATORS

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Reference: Tom Lancaster and Stephen J. Blundell, *Quantum Field Theory for the Gifted Amateur*, (Oxford University Press, 2014), Problem 13.4.

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In their section 13.3, L&B show that the projection of a four-vector along a given four-momentum p^{μ} is given by the longitudinal projection operator

$$P_L^{\mu\nu} = \frac{p^{\mu}p^{\nu}}{p^2} \tag{1}$$

The projection transverse to the momentum is then given by

$$P_T^{\mu\nu} = g^{\mu\nu} - \frac{p^{\mu}p^{\nu}}{p^2} \tag{2}$$

The Lagrangian density for a free electromagnetic field is

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} \tag{3}$$

$$= -\frac{1}{4}\left(\partial^{\mu}A^{\nu} - \partial^{\nu}A^{\mu}\right)\left(\partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu}\right) \tag{4}$$

$$= -\frac{1}{4}\left(\partial^{\mu}A^{\nu}\partial_{\mu}A_{\nu} + \partial^{\nu}A^{\mu}\partial_{\nu}A_{\mu} - \partial^{\nu}A^{\mu}\partial_{\mu}A_{\nu} - \partial^{\mu}A^{\nu}\partial_{\nu}A_{\mu}\right) \tag{5}$$

$$= -\frac{1}{4}\left(2\partial^{\mu}A^{\nu}\partial_{\mu}A_{\nu} - 2\partial^{\nu}A^{\mu}\partial_{\mu}A_{\nu}\right) \tag{6}$$

$$= -\frac{1}{2}\left(\partial^{\mu}A^{\nu}\partial_{\mu}A_{\nu} - \partial^{\nu}A^{\mu}\partial_{\mu}A_{\nu}\right) \tag{7}$$

To get the fourth line, we swapped $\mu \leftrightarrow \nu$ in the second and fourth terms in the third line, which is permissible since these indexes are summed.

To express $\mathcal{L}$ in the form given by L&B's equation 13.43, we need to use the old trick of adding in a total divergence to the Lagrangian. This is allowed since we're assuming that the total Lagrangian is the integral of $\mathcal{L}$ over all space and, using Gauss's theorem, the integral of a total divergence converts to a surface integral which goes to zero at infinity. In this case, we consider the term

$$K_\mu = \frac{1}{2}A^\nu\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right) \tag{8}$$

The divergence gives us

$$\partial^\mu K_\mu = \frac{1}{2}\partial^\mu A^\nu\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right) + \frac{1}{2}A^\nu\left(\partial^\mu\partial_\mu A_\nu - \partial_\nu\partial^\mu A_\mu\right) \tag{9}$$

We now observe that the first term in 9 is the negative of $\mathcal{L}$ as given in 7, so we can write

$$\mathcal{L} = \frac{1}{2}A^\nu\left(\partial^\mu\partial_\mu A_\nu - \partial_\nu\partial^\mu A_\mu\right) - \partial^\mu K_\mu \tag{10}$$

and since we can throw away any total divergence that appears in the Lagrangian density, we can simplify this to

$$\mathcal{L} = \frac{1}{2}A^\nu\left(\partial^\mu\partial_\mu A_\nu - \partial_\nu\partial^\mu A_\mu\right) \tag{11}$$

$$= \frac{1}{2}A^\nu\left(\partial^2 A_\nu - \partial_\nu\partial^\mu A_\mu\right) \tag{12}$$

To relate this to the projection operator P_T in 2, we need to note that $\mathcal{L}$ is given as a function of x (since all the fields A^μ are functions of x), not p, so we need to write P_T in spacetime coordinates, instead of momentum coordinates. The spacetime representation of p^μ is $i\partial^\mu$, so in these coordinates, we have

$$P_T^{\mu\nu} = g^{\mu\nu} - \frac{\left(i\partial^\mu\right)\left(i\partial^\nu\right)}{\left(i\partial\right)^2} \tag{13}$$

$$= g^{\mu\nu} - \frac{\partial^\mu\partial^\nu}{\partial^2} \tag{14}$$

The term containing the differential operators is assumed to act on whatever field is written to the right of P_T.

We can now write 12 as

$$\mathcal{L} = \frac{1}{2}A^{\nu}\left(\partial^2 A_{\nu} - \partial_{\nu}\partial_{\mu}A^{\mu}\right) \tag{15}$$

$$= \frac{1}{2}A_{\nu}\partial^2 A^{\nu} - \frac{1}{2}A^{\nu}\partial_{\nu}\partial_{\mu}A^{\mu} \tag{16}$$

$$= \frac{1}{2}A^{\mu}g_{\mu\nu}\partial^2 A^{\nu} - \frac{1}{2}A^{\mu}\partial_{\mu}\partial_{\nu}A^{\nu} \tag{17}$$

$$= \frac{1}{2}A^{\mu}\left(g_{\mu\nu} - \frac{\partial_{\mu}\partial_{\nu}}{\partial^2}\right)\partial^2 A^{\nu} \tag{18}$$

$$= \frac{1}{2}A^{\mu}P^{T}_{\mu\nu}\partial^2 A^{\nu} \tag{19}$$