ELECTROMAGNETISM: QUANTUM HAMILTONIAN IN COULOMB GAUGE

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Reference: Tom Lancaster and Stephen J. Blundell, *Quantum Field Theory for the Gifted Amateur*, (Oxford University Press, 2014), Problem 14.1.

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In classical electromagnetism, the Coulomb gauge is defined by the condition on the vector field:

$$\nabla\cdot\mathbf{A}=0 \tag{1}$$

The Coulomb gauge treated in L&B has the additional condition

$$A^{0}=0 \tag{2}$$

The Lagrangian for a free electromagnetic field is

$$\mathcal{L}=-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} \tag{3}$$

which, for the Coulomb gauge, gives a Hamiltonian density

$$\mathcal{H}=\frac{1}{2}\left(\mathbf{E}^{2}+\mathbf{B}^{2}\right) \tag{4}$$

In terms of the electromagnetic tensor, we have

$$E_{j}=F_{0j}=\partial_{0}A_{j}-\partial_{j}A_{0}=\partial_{0}A_{j} \tag{5}$$

The magnetic field components are given by

$$B_{1}=\partial_{2}A_{3}-\partial_{3}A_{2} \tag{6}$$
$$B_{2}=\partial_{3}A_{1}-\partial_{1}A_{3} \tag{7}$$
$$B_{3}=\partial_{1}A_{2}-\partial_{2}A_{1} \tag{8}$$

The mode expansion of the field A^{μ} is given by

$$A^{\mu}(x)=\int\frac{d^{3}p}{\sqrt{(2\pi)^{3}2E_{\mathbf{p}}}}\sum_{\lambda=1}^{2}\left[\epsilon_{\lambda}^{\mu}(p)\,a_{\lambda\mathbf{p}}e^{-ip\cdot x}+\epsilon_{\lambda}^{\mu*}(p)\,a_{\lambda\mathbf{p}}^{\dagger}e^{ip\cdot x}\right] \tag{9}$$

where the ϵ_{λ}^{μ} are the polarization vectors. In this case, there are only two ($\lambda = 1,2$) polarization vectors since electromagnetic waves are transverse (perpendicular to the direction of the momentum). Our goal is to plug this mode expansion into 4 and derive an expression for the total Hamiltonian in terms of creation and annihilation operators. This gets a bit messy but we can plow onwards and see how it goes.

In order to achieve this, we need a couple of properties of the polarization vectors. Since they are transverse, they are perpendicular to the momentum p so we have

$$p_{\mu}\epsilon_{\lambda}^{\mu}(p) = 0 \tag{10}$$

for both vectors $\lambda = 1,2$. We also assume they are normalized according to

$$g_{\mu\nu}\epsilon_{\lambda}^{\mu*}(p)\,\epsilon_{\kappa}^{\nu}(p) = -\delta_{\lambda\kappa} \tag{11}$$

In practice, since $A^0 = 0$, $\epsilon^0 = 0$ and the vectors are essentially three-dimensional, so we can write the condition as a simple dot product:

$$\boldsymbol{\epsilon}_{\lambda}^{*}\cdot\boldsymbol{\epsilon}_{\kappa} = \delta_{\lambda\kappa} \tag{12}$$

Actually, to make the derivation work, we need to assume

$$\boldsymbol{\epsilon}_{\lambda}\cdot\boldsymbol{\epsilon}_{\kappa} = \delta_{\lambda\kappa} \tag{13}$$

I've seen this condition specified in sources other than L&B's book, so hopefully it is correct.

First, we consider $\mathbf{E}^2 = E_1^2 + E_2^2 + E_3^2$. We have

$$E_j = \partial_0 A_j \tag{14}$$

$$= i\int\frac{d^3p\;E_{\mathbf{p}}}{\sqrt{(2\pi)^3\,2E_{\mathbf{p}}}}\sum_{\lambda=1}^{2}\left[-\epsilon_{\lambda}^{\mu}(p)\,a_{\lambda\mathbf{p}}e^{-ip\cdot x} + \epsilon_{\lambda}^{\mu*}(p)\,a_{\lambda\mathbf{p}}^{\dagger}e^{ip\cdot x}\right] \tag{15}$$

We need to square this and integrate the result over all space. The square will contain four terms, consisting if a term with two annihilation operators $a_{\lambda\mathbf{p}}$, one with two creation operators $a_{\lambda\mathbf{p}}^{\dagger}$, and two other terms each containing one annihilation and one creation operator. We look first at the term containing two annihilation operators. This is

$$\int d^3x\,(E_j)^2\Big|_{aa} = -\frac{1}{2}\int d^3x\,\frac{d^3p\;d^3q\;E_{\mathbf{p}}E_{\mathbf{q}}}{(2\pi)^3\sqrt{E_{\mathbf{p}}E_{\mathbf{q}}}}\sum_{\lambda,\kappa}\epsilon_{\lambda\mathbf{p}}^{j}\epsilon_{\kappa\mathbf{q}}^{j}a_{\lambda\mathbf{p}}a_{\kappa\mathbf{q}}e^{-i(p+q)\cdot x} \tag{16}$$

Integrating over d^3x introduces a delta function $\delta(\mathbf{p}+\mathbf{q})$ which allows us to do the integral over d^3q by setting $\mathbf{q}=-\mathbf{p}$, with the result (since $E_{\mathbf{p}}=E_{-\mathbf{p}}$)

$$\int d^3x\,(E_j)^2\bigg|_{aa} = -\frac{1}{2}\int d^3p\; E_{\mathbf{p}}\sum_{\lambda,\kappa}\epsilon^j_{\lambda\mathbf{p}}\epsilon^j_{\kappa-\mathbf{p}}a_{\lambda\mathbf{p}}a_{\kappa-\mathbf{p}}e^{-2iE_{\mathbf{p}}t} \tag{17}$$

Using 13 we can sum over j to get

$$\int d^3x\;\mathbf{E}^2\bigg|_{aa} = -\frac{1}{2}\int d^3p\; E_{\mathbf{p}}\sum_{\lambda}a_{\lambda\mathbf{p}}a_{\lambda-\mathbf{p}}e^{-2iE_{\mathbf{p}}t} \tag{18}$$

Now consider the component B_1 of the magnetic field. Plugging 9 into 6 we have

$$B_1 = i\int\frac{d^3p}{\sqrt{(2\pi)^3\,2E_{\mathbf{p}}}}\sum_{\lambda=1}^{2}\left[a_{\lambda\mathbf{p}}e^{-ip\cdot x}\left(p_2\epsilon^3_{\lambda\mathbf{p}}-p_3\epsilon^2_{\lambda\mathbf{p}}\right)-\right. \tag{19}$$
$$\left. a^\dagger_{\lambda\mathbf{p}}e^{ip\cdot x}\left(p_2\epsilon^{3*}_{\lambda\mathbf{p}}-p_3\epsilon^{2*}_{\lambda\mathbf{p}}\right)\right]$$

Again, when we square this and integrate over d^3x we will get four terms involving the above mentioned combinations of annihilation and creation operators. Consider the term containing two annihilation operators. I'll show the result after integrating over d^3x, and then using the $\delta(\mathbf{p}+\mathbf{q})$ to set $\mathbf{q}=-\mathbf{p}$:

Note that the superscripts on the ϵ label the component of the vector and are not exponents. Superscripts on everything else *are* exponents.

$$\int d^3x\,(B_1)^2\bigg|_{aa} = -\frac{1}{2}\int\frac{d^3p}{E_{\mathbf{p}}}\sum_{\lambda,\kappa}\left(p_2\epsilon^3_{\lambda\mathbf{p}}-p_3\epsilon^2_{\lambda\mathbf{p}}\right)\left(-p_2\epsilon^3_{\kappa-\mathbf{p}}+p_3\epsilon^2_{\kappa-\mathbf{p}}\right)a_{\lambda\mathbf{p}}a_{\kappa-\mathbf{p}}e^{-2iE_{\mathbf{p}}t} \tag{20}$$

$$= \frac{1}{2}\int\frac{d^3p}{E_{\mathbf{p}}}\sum_{\lambda,\kappa}\left[p_2^2\epsilon^3_{\lambda\mathbf{p}}\epsilon^3_{\kappa-\mathbf{p}}+p_3^2\epsilon^2_{\lambda\mathbf{p}}\epsilon^2_{\kappa-\mathbf{p}}-\right. \tag{21}$$
$$\left. p_2p_3\left(\epsilon^3_{\lambda\mathbf{p}}\epsilon^2_{\kappa-\mathbf{p}}+\epsilon^2_{\lambda\mathbf{p}}\epsilon^3_{\kappa-\mathbf{p}}\right)\right]a_{\lambda\mathbf{p}}a_{\kappa-\mathbf{p}}e^{-2iE_{\mathbf{p}}t}$$

We can get the results for the other two components of the magnetic field by permuting the indexes. We have

$$\int d^3x\,(B_2)^2\Big|_{aa} = \frac{1}{2}\int\frac{d^3p}{E_{\mathbf{p}}}\sum_{\lambda,\kappa}\Big[p_3^2\epsilon^1_{\lambda\mathbf{p}}\epsilon^1_{\kappa-\mathbf{p}} + p_1^2\epsilon^3_{\lambda\mathbf{p}}\epsilon^3_{\kappa-\mathbf{p}} - p_3p_1\left(\epsilon^1_{\lambda\mathbf{p}}\epsilon^3_{\kappa-\mathbf{p}} + \epsilon^3_{\lambda\mathbf{p}}\epsilon^1_{\kappa-\mathbf{p}}\right)\Big]a_{\lambda\mathbf{p}}a_{\kappa-\mathbf{p}}e^{-2iE_{\mathbf{p}}t} \tag{22}$$

$$\int d^3x\,(B_3)^2\Big|_{aa} = \frac{1}{2}\int\frac{d^3p}{E_{\mathbf{p}}}\sum_{\lambda,\kappa}\Big[p_1^2\epsilon^2_{\lambda\mathbf{p}}\epsilon^2_{\kappa-\mathbf{p}} + p_2^2\epsilon^1_{\lambda\mathbf{p}}\epsilon^1_{\kappa-\mathbf{p}} - p_1p_2\left(\epsilon^2_{\lambda\mathbf{p}}\epsilon^1_{\kappa-\mathbf{p}} + \epsilon^1_{\lambda\mathbf{p}}\epsilon^2_{\kappa-\mathbf{p}}\right)\Big]a_{\lambda\mathbf{p}}a_{\kappa-\mathbf{p}}e^{-2iE_{\mathbf{p}}t} \tag{23}$$

We can now use 13 to write

$$\epsilon^2_{\lambda\mathbf{p}}\epsilon^2_{\kappa\mathbf{p}} + \epsilon^3_{\lambda\mathbf{p}}\epsilon^3_{\kappa\mathbf{p}} = \delta_{\lambda\kappa} - \epsilon^1_{\lambda\mathbf{p}}\epsilon^1_{\kappa\mathbf{p}} \tag{24}$$

We must also assume that $\epsilon_{\lambda\mathbf{p}} = \epsilon_{\lambda-\mathbf{p}}$, which seems reasonable, since reversing the direction of the momentum won't affect vectors that are transverse to the momentum.

When we add up 21, 22 and 23, we will get a term

$$p_1^2\left(\epsilon^2_{\lambda\mathbf{p}}\epsilon^2_{\kappa-\mathbf{p}} + \epsilon^3_{\lambda\mathbf{p}}\epsilon^3_{\kappa-\mathbf{p}}\right) = p_1^2\left(\delta_{\lambda\kappa} - \epsilon^1_{\lambda\mathbf{p}}\epsilon^1_{\kappa\mathbf{p}}\right) \tag{25}$$

Similar terms arise for p_2^2 and p_3^2 so we get

$$\sum_{j=1}^{3}\int d^3x\,(B_j)^2\Big|_{aa} = \frac{1}{2}\int\frac{d^3p}{E_{\mathbf{p}}}\sum_{\lambda,\kappa}\Big[\left(p_1^2+p_2^2+p_3^2\right)\delta_{\lambda\kappa} - p_1^2\epsilon^1_{\lambda\mathbf{p}}\epsilon^1_{\kappa\mathbf{p}} - p_2^2\epsilon^2_{\lambda\mathbf{p}}\epsilon^2_{\kappa\mathbf{p}} - p_3^2\epsilon^3_{\lambda\mathbf{p}}\epsilon^3_{\kappa\mathbf{p}} - \text{c.t.}\Big]a_{\lambda\mathbf{p}}a_{\kappa-\mathbf{p}}e^{-2iE_{\mathbf{p}}t} \tag{26}$$

$$= \frac{1}{2}\int\frac{d^3p}{E_{\mathbf{p}}}\sum_{\lambda,\kappa}\Big[E_{\mathbf{p}}^2\delta_{\lambda\kappa} - p_1^2\epsilon^1_{\lambda\mathbf{p}}\epsilon^1_{\kappa\mathbf{p}} - p_2^2\epsilon^2_{\lambda\mathbf{p}}\epsilon^2_{\kappa\mathbf{p}} - p_3^2\epsilon^3_{\lambda\mathbf{p}}\epsilon^3_{\kappa\mathbf{p}} - \text{c.t.}\Big]a_{\lambda\mathbf{p}}a_{\kappa-\mathbf{p}}e^{-2iE_{\mathbf{p}}t} \tag{27}$$

where 'c.t.' stands for 'cross terms' which are the terms in 21, 22 and 23 that involve products of two different components of momentum.

Now we finally observe that, from 10, we can write

$$\left(p_\mu\epsilon^\mu_\lambda\right)\left(p_\mu\epsilon^\mu_\kappa\right) = \left(p_1\epsilon^1_\lambda + p_2\epsilon^2_\lambda + p_3\epsilon^3_\lambda\right)\left(p_1\epsilon^1_\kappa + p_2\epsilon^2_\kappa + p_3\epsilon^3_\kappa\right) \tag{28}$$

When we multiply the RHS out we find that it is equal to

$$\text{RHS} = p_1^2\epsilon^1_{\lambda\mathbf{p}}\epsilon^1_{\kappa\mathbf{p}} + p_2^2\epsilon^2_{\lambda\mathbf{p}}\epsilon^2_{\kappa\mathbf{p}} + p_3^2\epsilon^3_{\lambda\mathbf{p}}\epsilon^3_{\kappa\mathbf{p}} + \text{c.t.} \tag{29}$$

Thus this combination of terms is zero by the assumption 10. We are left with

$$\sum_{j=1}^{3}\int d^3x\,(B_j)^2\bigg|_{aa} = \frac{1}{2}\int\frac{d^3p}{E_{\mathbf{p}}}\sum_{\lambda,\kappa}E_{\mathbf{p}}^2\delta_{\lambda\kappa}a_{\lambda\mathbf{p}}a_{\kappa-\mathbf{p}}e^{-2iE_{\mathbf{p}}t} \tag{30}$$

$$= \frac{1}{2}\int d^3p\;E_{\mathbf{p}}\sum_{\lambda}a_{\lambda\mathbf{p}}a_{\lambda-\mathbf{p}}e^{-2iE_{\mathbf{p}}t} \tag{31}$$

which exactly cancels 18. Thus the term involving two annihilation operators is zero.

We can do a similar calculation for the term involving two creation operators to find that it too is zero.

For the terms involving one annihilation and one creation operator, the calculation is similar. The main difference is that integrating over d^3x gives us a delta function $\delta(\mathbf{p}-\mathbf{q})$, so now we set $\mathbf{q}=\mathbf{p}$ rather than $\mathbf{q}=-\mathbf{p}$, which results in the exponential factor becoming 1 and also in the contributions from the electric and magnetic fields having the same sign rather than opposite signs. We end up with

$$\sum_{j=1}^{3}\int d^3x\,(E_j)^2\bigg|_{aa^\dagger} = \frac{1}{2}\int d^3p\;E_{\mathbf{p}}\sum_{\lambda}a_{\lambda\mathbf{p}}a_{\lambda\mathbf{p}}^\dagger \tag{32}$$

$$\sum_{j=1}^{3}\int d^3x\,(B_j)^2\bigg|_{aa^\dagger} = \frac{1}{2}\int d^3p\;E_{\mathbf{p}}\sum_{\lambda}a_{\lambda\mathbf{p}}a_{\lambda\mathbf{p}}^\dagger \tag{33}$$

Finally for the other such term, we have

$$\sum_{j=1}^{3}\int d^3x\,(E_j)^2\bigg|_{a^\dagger a} = \frac{1}{2}\int d^3p\;E_{\mathbf{p}}\sum_{\lambda}a_{\lambda\mathbf{p}}^\dagger a_{\lambda\mathbf{p}} \tag{34}$$

$$\sum_{j=1}^{3}\int d^3x\,(B_j)^2\bigg|_{a^\dagger a} = \frac{1}{2}\int d^3p\;E_{\mathbf{p}}\sum_{\lambda}a_{\lambda\mathbf{p}}^\dagger a_{\lambda\mathbf{p}} \tag{35}$$

Adding everything up we have

$$H = \int d^3x\, \mathcal{H} \tag{36}$$

$$= \frac{1}{2}\int \left(\mathbf{E}^2 + \mathbf{B}^2\right) \tag{37}$$

$$= \frac{1}{2}\left[\int d^3p\; E_{\mathbf{p}} \sum_{\lambda} a_{\lambda\mathbf{p}} a^{\dagger}_{\lambda\mathbf{p}} + \int d^3p\; E_{\mathbf{p}} \sum_{\lambda} a^{\dagger}_{\lambda\mathbf{p}} a_{\lambda\mathbf{p}}\right] \tag{38}$$

$$= \int d^3p\; E_{\mathbf{p}} \sum_{\lambda} a^{\dagger}_{\lambda\mathbf{p}} a_{\lambda\mathbf{p}} \tag{39}$$

where to get the last line we applied normal ordering to the third line.

PINGBACKS

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