ROTATION OPERATOR FOR SPINORS

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Reference: Tom Lancaster and Stephen J. Blundell, *Quantum Field Theory for the Gifted Amateur*, (Oxford University Press, 2014), Problem 15.3.

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The rotation operator for spinors can be written as

$$\mathbf{R}(\hat{\mathbf{n}},\theta)=e^{-i\theta(\boldsymbol{\sigma}\cdot\mathbf{n})/2} \tag{1}$$

where $\boldsymbol{\sigma}$ is the 3-component vector of the Pauli spin matrices, θ is the angle of rotation and $\hat{\mathbf{n}}$ is a unit vector along the axis of rotation. The exponential in 1 can be expanded in a power series with the result

$$\mathbf{R}(\hat{\mathbf{n}},\theta)=I\cos\frac{\theta}{2}-i\boldsymbol{\sigma}\cdot\hat{\mathbf{n}}\sin\frac{\theta}{2} \tag{2}$$

If we multiply out the dot product, we have

$$\boldsymbol{\sigma}\cdot\hat{\mathbf{n}}=\left[\begin{array}{cc} n_z & n_x-in_y\\ n_x+in_y & -n_z\end{array}\right] \tag{3}$$

For rotations about the three coordinates axes, we have therefore

$$\mathbf{R}(\hat{\mathbf{x}},\theta)=I\cos\frac{\theta}{2}-i\left[\begin{array}{cc} 0 & 1\\ 1 & 0\end{array}\right]\sin\frac{\theta}{2} \tag{4}$$

$$=\left[\begin{array}{cc} \cos\frac{\theta}{2} & -i\sin\frac{\theta}{2}\\ -i\sin\frac{\theta}{2} & \cos\frac{\theta}{2}\end{array}\right] \tag{5}$$

$$\mathbf{R}(\hat{\mathbf{y}},\theta)=I\cos\frac{\theta}{2}-i\left[\begin{array}{cc} 0 & -i\\ i & 0\end{array}\right]\sin\frac{\theta}{2} \tag{6}$$

$$=\left[\begin{array}{cc} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2}\\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2}\end{array}\right] \tag{7}$$

$$\mathbf{R}(\hat{\mathbf{z}},\theta)=I\cos\frac{\theta}{2}-i\left[\begin{array}{cc} 1 & 0\\ 0 & -1\end{array}\right]\sin\frac{\theta}{2} \tag{8}$$

$$=\left[\begin{array}{cc} \cos\frac{\theta}{2}-i\sin\frac{\theta}{2} & 0\\ 0 & \cos\frac{\theta}{2}+i\sin\frac{\theta}{2}\end{array}\right] \tag{9}$$

$$=\left[\begin{array}{cc} e^{-i\theta/2} & 0\\ 0 & e^{i\theta/2}\end{array}\right] \tag{10}$$