GREEN FUNCTIONS IN THE ENERGY DOMAIN

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Reference: Tom Lancaster and Stephen J. Blundell, *Quantum Field Theory for the Gifted Amateur*, (Oxford University Press, 2014), Problem 16.2.

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The Green function for the Schrödinger equation has the form

$$G^{+}(x,t_x,y,t_y)=\theta(t_x-t_y)\sum_n\phi_n(x)\phi_n^*(y)e^{-iE_n(t_x-t_y)} \tag{1}$$

where ϕ_n is an eigenfunction of the hamiltonian H with energy E_n. Without loss of generality, we can take the start time to be $t_y=0$ and then replace t_x by t to get

$$G^{+}(x,t,y)=\theta(t)\sum_n\phi_n(x)\phi_n^*(y)e^{-iE_nt} \tag{2}$$

To get G^+ in the space-energy domain, we do a Fourier transform over the time t. We have

$$G^{+}(x,y,E)=\int dt\;e^{iEt}G^{+}(x,t,y) \tag{3}$$

$$=\int dt\;e^{iEt}\theta(t)\sum_n\phi_n(x)\phi_n^*(y)e^{-iE_nt} \tag{4}$$

$$=\int_0^{\infty}dt\;e^{iEt}\sum_n\phi_n(x)\phi_n^*(y)e^{-iE_nt} \tag{5}$$

where we've introduced the limits on the integral in the last line by using the step function $\theta(t)$, which is zero for $t<0$. Since t occurs only as part of the imaginary exponent, the integral doesn't converge, so we can introduce a damping factor $e^{i\epsilon}$ where ϵ is a positive infinitesimal quantity. This gives

$$G^{+}(x,y,E)=\int_{0}^{\infty}dt\,\sum_{n}\phi_{n}(x)\,\phi_{n}^{*}(y)\,e^{it(E-E_{n}+i\epsilon)} \tag{6}$$

$$=\frac{1}{i}\sum_{n}\phi_{n}(x)\,\phi_{n}^{*}(y)\,\frac{e^{it(E-E_{n}+i\epsilon)}}{E-E_{n}+i\epsilon}\Bigg|_{0}^{\infty} \tag{7}$$

$$=\sum_{n}\phi_{n}(x)\,\phi_{n}^{*}(y)\,\frac{i}{E-E_{n}+i\epsilon} \tag{8}$$

To get G^{+} in the momentum-energy domain, we start with L&B's equation 16.36 giving $G^{+}(p,t)$ (again, we're taking $t_{y}=0$ and $t_{x}=t$):

$$G^{+}(p,t)=\theta(t)\,e^{-iE_{p}t} \tag{9}$$

Using the formula (see L&B's Example B.5 in the appendix for details of the contour integration used to derive this):

$$\theta(t)=i\int_{-\infty}^{\infty}\frac{dz}{2\pi}\frac{e^{-izt}}{z+i\epsilon} \tag{10}$$

we can write 9 as

$$G^{+}(p,t)=i\int_{-\infty}^{\infty}\frac{dz}{2\pi}\frac{e^{-izt}}{z+i\epsilon}e^{-iE_{p}t} \tag{11}$$

Taking the Fourier transform over t, we have

$$G^{+}(p,E)=\int_{-\infty}^{\infty}dt\;e^{iEt}G^{+}(p,t) \tag{12}$$

$$=\frac{i}{2\pi}\int_{-\infty}^{\infty}\frac{dz}{z+i\epsilon}\int_{-\infty}^{\infty}dt\;e^{-i(z+E_{p}-E)t} \tag{13}$$

$$=i\int_{-\infty}^{\infty}\frac{dz}{z+i\epsilon}\delta(z+E_{p}-E) \tag{14}$$

$$=\frac{i}{E-E_{p}+i\epsilon} \tag{15}$$