GREEN FUNCTION FOR A FORCED HARMONIC OSCILLATOR

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Reference: Tom Lancaster and Stephen J. Blundell, *Quantum Field Theory for the Gifted Amateur*, (Oxford University Press, 2014), Problem 16.3.

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We've looked at using a Green function to solve the forced harmonic oscillator before, but in that case the forcing function was a delta function. Here we'll look at solving the problem a bit more generally.

Suppose we have a forced harmonic oscillator whose amplitude A obeys the differential equation

$$m\frac{\partial^2}{\partial t^2}A(t-u)+m\omega_0^2 A(t-u)=\tilde{F}(\omega)e^{-i\omega(t-u)} \tag{1}$$

We can show by direct substitution that the solution is

$$A(t-u)=-\frac{\tilde{F}(\omega)}{m}\frac{e^{-i\omega(t-u)}}{\omega^2-\omega_0^2}+B(t) \tag{2}$$

where $B(t)$ is a solution to the homogeneous equation, that is:

$$m\ddot{B}(t)+m\omega_0^2 B(t)=0 \tag{3}$$

We have

$$\frac{\partial^2}{\partial t^2}A(t-u)=\frac{\tilde{F}(\omega)}{m}\frac{e^{-i\omega(t-u)}}{\omega^2-\omega_0^2}\omega^2+\ddot{B}(t) \tag{4}$$

Substituting these two results into 1 we have

$$m\frac{\partial^2}{\partial t^2}A(t-u)+m\omega_0^2 A(t-u)=m\frac{\tilde{F}(\omega)}{m}\frac{e^{-i\omega(t-u)}}{\omega^2-\omega_0^2}\left(\omega^2-\omega_0^2\right)+m\ddot{B}+m\omega_0^2 B \tag{5}$$

$$=\tilde{F}(\omega)e^{-i\omega(t-u)}+0 \tag{6}$$

The Green function for 1 is given by

$$m\left(\partial_t^2+\omega_0^2\right)G(t,u)=\delta(t-u) \tag{7}$$

We can write the delta function as an integral in the usual way to get

$$m\left(\partial_t^2+\omega_0^2\right)G(t,u)=\frac{1}{2\pi}\int d\omega e^{-i\omega(t-u)} \tag{8}$$

We therefore have

$$m\left(\partial_t^2+\omega_0^2\right)\left[\frac{1}{2\pi}\int d\omega e^{-i\omega(t-u)}\right]=\frac{m}{2\pi}\int d\omega\ \left(\omega_0^2-\omega^2\right)e^{-i\omega(t-u)} \tag{9}$$

Therefore if we take

$$G(t,u)=-\frac{1}{2\pi m}\int d\omega\frac{e^{-i\omega(t-u)}}{\omega^2-\omega_0^2} \tag{10}$$

this satisfies 7. To get the general solution for the Green function, we can add on $B(t)$, since because of 3, this will add zero to the LHS of 7. Thus the general Green function is given by

$$G(t,u)=-\frac{1}{2\pi m}\int d\omega\frac{e^{-i\omega(t-u)}}{\omega^2-\omega_0^2}+B(t) \tag{11}$$

We can solve 7 using Laplace transforms. If you're unfamiliar with Laplace transforms, a good introduction can be found (at the time of writing - 3 Jun 2019) here. Most textbooks on differential equations or advanced calculus should also describe how to define and use them.

In practice, it's easiest to use a table of Laplace transforms rather than work them all out from scratch, so we'll do that here. To solve 7 we can think of this equation as an ordinary differential equation with independent variable t and constant parameter u. The Laplace transform of a second derivative is given by

$$\mathcal{L}\left(\partial_t^2 G(t,u)\right)=s^2\Gamma(s)-sG(0,u)-\dot{G}(0,u) \tag{12}$$

where $\Gamma(s)$ is the Laplace transform of $G(t,u)$. The variable s is the usual Laplace transform parameter, which will disappear when we convert the answer back at the end.

The given initial conditions in the problem are $G(0,u)=\dot{G}(0,u)=0$, so we have

$$\mathcal{L}\left(\partial_t^2 G(t,u)\right)=s^2\Gamma(s) \tag{13}$$

On the RHS of 7, we see from a table that the Laplace transform of a delta function is

$$\mathcal{L}(\delta(t-u))=e^{-us} \tag{14}$$

Putting all this together we get the Laplace transform of 7:

$$m\left(s^2+\omega_0^2\right)\Gamma(s)=e^{-us} \tag{15}$$

$$\Gamma(s)=\frac{e^{-us}}{m\left(s^2+\omega_0^2\right)} \tag{16}$$

Again from a table, we see that we can write this as

$$\frac{e^{-us}}{m\left(s^2+\omega_0^2\right)}=e^{-us}H(s) \tag{17}$$

where

$$H(s)=\frac{1}{m\left(s^2+\omega_0^2\right)} \tag{18}$$

An exponential e^{-us} multiplying a Laplace transform indicates an original function that is delayed by the amount u. That is, if $h(t)$ is the inverse transform of $H(s)$, then the inverse transform of $e^{-us}H(s)$ is $h(t-u)$.

From tables, the inverse transform of $H(s)$ is given by

$$h(t)=\frac{1}{m\omega_0}\sin\omega_0 t \tag{19}$$

Thus the inverse transform of 16 is

$$G(t,u)=\frac{1}{m\omega_0}\sin\omega_0(t-u) \tag{20}$$

Now suppose we have an explicit form of the forcing function given by

$$f(t)=F_0\sin\omega_0 t \tag{21}$$

The differential equation for this oscillator is therefore

$$m\frac{\partial^2}{\partial t^2}A(t)+m\omega_0^2A(t)=F_0\sin\omega_0 t \tag{22}$$

With the Green function given by 20, we can find $A(t)$ by doing the integral

$$A(t)=\int_0^t G(t,u)f(u)\,du \tag{23}$$

$$=\frac{F_0}{m\omega_0}\int_0^t du\ \sin\omega_0(t-u)\sin\omega_0 u \tag{24}$$

$$=\frac{F_0}{2m\omega_0^2}\left(\sin\omega_0 t-\omega_0 t\cos\omega_0 t\right) \tag{25}$$

I used Maple to do the integral. If you want to do it by hand, you'll need to expand $\sin\omega_0(t-u)$ and then use some double angle formulas from trigonometry.