GREEN FUNCTION FOR THE WAVE EQUATION

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Reference: Tom Lancaster and Stephen J. Blundell, *Quantum Field Theory for the Gifted Amateur*, (Oxford University Press, 2014), Problem 16.4.

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In their example 16.2, L&B consider the Green function for a version of the wave equation, where the Green function is defined by

$$\left(\nabla^2+\mathbf{k}^2\right)G_{\mathbf{k}}\left(\mathbf{x},\mathbf{u}\right)=\delta^{(3)}\left(\mathbf{x}-\mathbf{u}\right) \tag{1}$$

Here, we'll consider the special case $\mathbf{u}=0$, so we have

$$\left(\nabla^2+\mathbf{k}^2\right)G_{\mathbf{k}}\left(\mathbf{x}\right)=\delta^{(3)}\left(\mathbf{x}\right) \tag{2}$$

We can Fourier transform this to the momentum domain.We have

$$G_{\mathbf{k}}\left(\mathbf{x}\right)=\int d^3q\, e^{i\mathbf{q}\cdot\mathbf{x}}G_{\mathbf{k}}\left(q\right) \tag{3}$$

Applying the wave operator to this gives us

$$\left(\nabla^2+\mathbf{k}^2\right)G_{\mathbf{k}}\left(\mathbf{x}\right)=\int d^3q\,\left(\nabla^2+\mathbf{k}^2\right)e^{i\mathbf{q}\cdot\mathbf{x}}G_{\mathbf{k}}\left(q\right) \tag{4}$$

$$=\int d^3q\,\left(\mathbf{k}^2-\mathbf{q}^2\right)e^{i\mathbf{q}\cdot\mathbf{x}}G_{\mathbf{k}}\left(q\right) \tag{5}$$

In order for this integral to be the delta function $\delta^{(3)}\left(\mathbf{x}\right)$, we must therefore have

$$G_{\mathbf{k}}\left(q\right)=\frac{1}{\left(2\pi\right)^3}\frac{1}{\mathbf{k}^2-\mathbf{q}^2} \tag{6}$$

This differs from L&B's equation 16.46 by the factor of $(2\pi)^3$ but I suppose it depends where you put this factor when defining the Green function.

In L&B's equation 16.12 (with $\mathbf{u}=0$) they give one solution for the Green function:

$$G_{\mathbf{k}}^{+}\left(\mathbf{x}\right)=-\frac{e^{ikx}}{4\pi x} \tag{7}$$

[To simplify the notation, I've used ordinary (non-bold) symbols to denote the magnitudes of vectors, so that $k\equiv|\mathbf{k}|$ and so on. Note that the kx term in the exponent is simply the product of the magnitudes of $\mathbf{k}$ and $\mathbf{x}$, and *not* the dot product!] We now want to demonstrate that this solution corresponds to a momentum domain Green function of the form

$$\tilde{G}_{\mathbf{k}}^{+}(\mathbf{q}) = \frac{1}{k^2 - q^2 + i\epsilon} \tag{8}$$

To do this, we introduce a damping factor ϵ into 7 so we have instead

$$G_{\mathbf{k}}^{+}(\mathbf{x}) = -\frac{e^{ikx-\epsilon x}}{4\pi x} \tag{9}$$

The Fourier transform of this is

$$\tilde{G}_{\mathbf{k}}^{+}(\mathbf{q}) = -\int d^3x\, \frac{e^{i\mathbf{q}\cdot\mathbf{x}+ikx-\epsilon x}}{4\pi x} \tag{10}$$

To do this integral, we'll use spherical coordinates, so we have

$$\tilde{G}_{\mathbf{k}}^{+}(\mathbf{q}) = -\int d\phi d\theta dx\, x^2 \sin\theta \frac{e^{iqx\cos\theta+ikx-\epsilon x}}{4\pi x} \tag{11}$$

$$= -\frac{1}{2}\int_0^\infty dx\, x e^{i(kx+i\epsilon x)} \int_0^\pi \sin\theta e^{iqx\cos\theta} \tag{12}$$

$$= \frac{1}{2}\int_0^\infty dx\, x e^{i(kx+i\epsilon x)} \frac{1}{iqx}\left(e^{-iqx} - e^{iqx}\right) \tag{13}$$

$$= \frac{1}{2iq}\int_0^\infty dx\, \left(e^{i(kx-qx+i\epsilon x)} - e^{i(kx+qx+i\epsilon x)}\right) \tag{14}$$

$$= -\frac{1}{2iq}\left[\frac{1}{ik-iq-\epsilon} - \frac{1}{ik+iq-\epsilon}\right] \tag{15}$$

$$= -\frac{1}{2q}\left[\frac{1}{-k+q-i\epsilon} - \frac{1}{-k-q-i\epsilon}\right] \tag{16}$$

$$= -\frac{1}{2q}\frac{-k-q-i\epsilon-(-k+q-i\epsilon)}{(k-q+i\epsilon)(k+q+i\epsilon)} \tag{17}$$

$$= \frac{1}{k^2-q^2+2ik\epsilon-\epsilon^2} \tag{18}$$

Since we're taking the damping factor ϵ to be infinitesimal, we can ignore the ϵ^2 in the denominator and call the $2ik\epsilon$ term just $i\epsilon$. Thus to order ϵ we have

$$\tilde{G}_{\mathbf{k}}^{+}(\mathbf{q}) = \frac{1}{k^2 - q^2 + i\epsilon} \tag{19}$$

L&B now ask us to reverse the procedure and re-derive 9 from 19. To do this we take the inverse Fourier transform of 19. My understanding of the inverse transform is that we need to multiply by an exponential with the opposite sign in the exponent, but if we do this, the answer doesn't work out, so perhaps I'm missing something. Anyway, to proceed, we have

The factor of $(2\pi)^3$ is required in either the forward or inverse Fourier transform, and it seems L&B's convention is put it in the inverse transform.

$$G^+_{\mathbf{k}}(\mathbf{x}) = \frac{1}{(2\pi)^3}\int d^3q\, e^{i\mathbf{q}\cdot\mathbf{x}}\tilde{G}^+_{\mathbf{k}}(\mathbf{q}) \tag{20}$$

$$= \frac{1}{(2\pi)^3}\int d^3q\, \frac{e^{i\mathbf{q}\cdot\mathbf{x}}}{k^2-q^2+i\epsilon} \tag{21}$$

$$= \frac{1}{4\pi^2}\int_0^\infty dq \frac{q^2}{k^2-q^2+i\epsilon}\int_0^\pi d\theta\, \sin\theta e^{iqx\cos\theta} \tag{22}$$

$$= -\frac{1}{4\pi^2}\int_0^\infty dq \frac{q}{q^2-k^2-i\epsilon}\frac{e^{iqx}-e^{-iqx}}{ix} \tag{23}$$

To do this integral, we need to use contour integration. First, we note that the integrand in 23 is even, so that replacing $q \to -q$ leaves it unchanged. Thus we can extend the lower limit of integration from 0 to $-\infty$ and divide by 2, since the contributions from the lower and upper ranges of integration are the same. We have

$$G^+_{\mathbf{k}}(\mathbf{x}) = -\frac{1}{8\pi^2 ix}\int_{-\infty}^\infty dq \frac{q}{q^2-k^2-i\epsilon}\left(e^{iqx}-e^{-iqx}\right) \tag{24}$$

We can now split the integral into its two component parts, so that

$$G^+_{\mathbf{k}}(\mathbf{x}) = -\frac{1}{8\pi^2 ix}(I_1 - I_2) \tag{25}$$

with

$$I_1 \equiv \int_{-\infty}^\infty dq \frac{qe^{iqx}}{q^2-k^2-i\epsilon} \tag{26}$$

$$I_2 \equiv \int_{-\infty}^\infty dq \frac{qe^{-iqx}}{q^2-k^2-i\epsilon} \tag{27}$$

Consider I_1. The integrand has poles where the denominator has zeroes, which occur at

$$q = \pm\sqrt{k^2+i\epsilon} \tag{28}$$

Since ϵ is infinitesimal, we can approximate this by

$$q = \pm k\sqrt{1+\frac{i\epsilon}{k^2}} \tag{29}$$

$$\approx \pm k\left(1+\frac{i\epsilon}{2k^2}\right) \tag{30}$$

$$= \pm\left(k+\frac{i\epsilon}{2k}\right) \tag{31}$$

We can now integrate I_1 over a contour consisting of a semi-circle with an edge along the real axis and an arc in the upper half plane. We traverse the semi-circle in a counter-clockwise direction so that the path along the real axis runs in the positive direction. In the upper half plane, we can write $q = Re^{i\phi}$ where R is the radius of the arc and ϕ runs from 0 to π. In the upper half plane, the imaginary part of q is always positive, so as $R \to \infty$, $e^{iqx} \to 0$. For large R, the quantity $\frac{q\,dq}{q^2-k^2-i\epsilon}$ goes as $R^2/R^2 = 1$, so integrand in 26 is zero along the arc. As a result, from Cauchy's theorem, we have

$$I_1 = \int_{-\infty}^{\infty} dq \frac{qe^{iqx}}{q^2-k^2-i\epsilon} = 2\pi i \sum_{\text{poles}} (\text{residues within the semi-circle}) \tag{32}$$

From 31, the only pole within the contour is at $q = +\left(k+\frac{i\epsilon}{2k}\right)$ and the residue of the integrand in 26 is

$$\text{Res} = \lim_{q\to k+\frac{i\epsilon}{2k}} \left(q-k-\frac{i\epsilon}{2k}\right)\frac{qe^{iqx}}{q^2-k^2-i\epsilon} \tag{33}$$

$$= \lim_{q\to k+\frac{i\epsilon}{2k}} \left(q-k-\frac{i\epsilon}{2k}\right)\frac{qe^{iqx}}{\left(q-k-\frac{i\epsilon}{2k}\right)\left(q+k+\frac{i\epsilon}{2k}\right)} \tag{34}$$

$$= \frac{\left(k+\frac{i\epsilon}{2k}\right)}{2\left(k+\frac{i\epsilon}{2k}\right)} e^{i\left(k+\frac{i\epsilon}{2k}\right)x} \tag{35}$$

$$= \frac{1}{2}e^{i\left(k+\frac{i\epsilon}{2k}\right)x} \tag{36}$$

Putting this into 32 we have

$$I_1 = \pi i e^{i\left(k+\frac{i\epsilon}{2k}\right)x} \tag{37}$$

The argument for I_2 works the same way, except that we now must use a semi-circle in the lower half plane to make the integral along the arc vanish.

This requires traversing the semi-circle in a clockwise direction, so the integral has the opposite sign to I_1. This time, the only pole within the contour is at $q = -\left(k + \frac{i\epsilon}{2k}\right)$, and the residue comes out to be the same as in 36. Thus we get

$$I_2 = -\pi i e^{i\left(k+\frac{i\epsilon}{2k}\right)x} \tag{38}$$

Plugging these into 24 we get for the final result:

$$G_{\mathbf{k}}^{+}(\mathbf{x}) = -\frac{1}{8\pi^2 ix}\left(\pi i e^{i\left(k+\frac{i\epsilon}{2k}\right)x} + \pi i e^{i\left(k+\frac{i\epsilon}{2k}\right)x}\right) \tag{39}$$

$$= -\frac{e^{i\left(k+\frac{i\epsilon}{2k}\right)x}}{4\pi x} \tag{40}$$

Taking the limit $\epsilon \to 0$ gives

$$G_{\mathbf{k}}^{+}(\mathbf{x}) = -\frac{e^{ikx}}{4\pi x} \tag{41}$$

Although I don't fully understand the Green function for the wave equation, L&B state that the solution 7 is for an outgoing spherical wave. For an incoming wave, I'd guess that the direction of the momentum vector **k** would be reversed, so the Green function in this case would be

$$G_{\mathbf{k}}^{+}(\mathbf{x}) = -\frac{e^{-ikx}}{4\pi x} \tag{42}$$