

FIELD OPERATORS FOR THE INFINITE SQUARE WELL

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Reference: Tom Lancaster and Stephen J. Blundell, *Quantum Field Theory for the Gifted Amateur*, (Oxford University Press, 2014), Section 4.1.

The creation and annihilation operators $\hat{a}_{\mathbf{p}}^\dagger$ and $\hat{a}_{\mathbf{p}}$ create and annihilate a particle in a specific momentum state so, because of the uncertainty principle, such states are completely unlocalized in position. We can construct analogous operators that create and annihilate a particle at a specific position. Such operators are called *field operators*. As you would expect, a field operator, operating at a precise position, must be completely unlocalized in momentum.

The creation and annihilation field operators for a particle in a 3-d infinite square well are defined as

$$\hat{\psi}^\dagger(\mathbf{x}) \equiv \frac{1}{\sqrt{\mathcal{V}}} \sum_{\mathbf{p}} \hat{a}_{\mathbf{p}}^\dagger e^{-i\mathbf{p}\cdot\mathbf{x}} \quad (1)$$

$$\hat{\psi}(\mathbf{x}) \equiv \frac{1}{\sqrt{\mathcal{V}}} \sum_{\mathbf{p}} \hat{a}_{\mathbf{p}} e^{i\mathbf{p}\cdot\mathbf{x}} \quad (2)$$

where \mathcal{V} is the volume of the square well.

To see that they actually do create and annihilate a particle at position \mathbf{x} we'll have a look at their effect when operating on particular states.

First, we'll apply $\hat{\psi}^\dagger(\mathbf{x})$ to the vacuum state.

$$\hat{\psi}^\dagger(\mathbf{x})|0\rangle = \frac{1}{\sqrt{\mathcal{V}}} \sum_{\mathbf{p}} e^{-i\mathbf{p}\cdot\mathbf{x}} \hat{a}_{\mathbf{p}}^\dagger |0\rangle \quad (3)$$

$$= \frac{1}{\sqrt{\mathcal{V}}} \sum_{\mathbf{p}} e^{-i\mathbf{p}\cdot\mathbf{x}} |\mathbf{p}\rangle \quad (4)$$

We can now insert the unit operator in the form

$$1 = \int d^3\mathbf{y} |\mathbf{y}\rangle \langle \mathbf{y}| \quad (5)$$

and we get

$$\hat{\psi}^\dagger(\mathbf{x})|0\rangle = \frac{1}{\sqrt{\mathcal{V}}} \int d^3\mathbf{y} \sum_{\mathbf{p}} e^{-i\mathbf{p}\cdot\mathbf{x}} |\mathbf{y}\rangle \langle \mathbf{y}|\mathbf{p}\rangle \quad (6)$$

$$= \frac{1}{\sqrt{\mathcal{V}}} \int d^3\mathbf{y} \sum_{\mathbf{p}} e^{-i\mathbf{p}\cdot\mathbf{x}} |\mathbf{y}\rangle \left[\frac{1}{\sqrt{\mathcal{V}}} e^{i\mathbf{p}\cdot\mathbf{y}} \right] \quad (7)$$

$$= \int d^3\mathbf{y} \left[\frac{1}{\mathcal{V}} \sum_{\mathbf{p}} e^{i\mathbf{p}\cdot(\mathbf{y}-\mathbf{x})} \right] |\mathbf{y}\rangle \quad (8)$$

$$= \int d^3\mathbf{y} \delta^{(3)}(\mathbf{y}-\mathbf{x}) |\mathbf{y}\rangle \quad (9)$$

$$= |\mathbf{x}\rangle \quad (10)$$

So the creation operator $\hat{\psi}^\dagger(\mathbf{x})$ operating on the vacuum state creates a single particle at position \mathbf{x} .

We can now try the annihilation operator $\hat{\psi}(\mathbf{y})$ operating on a one-particle state $|\mathbf{x}\rangle$. We get

$$\hat{\psi}(\mathbf{y})|\mathbf{x}\rangle = \frac{1}{\sqrt{\mathcal{V}}} \sum_{\mathbf{p}} \hat{a}_{\mathbf{p}} e^{i\mathbf{p}\cdot\mathbf{y}} |\mathbf{x}\rangle \quad (11)$$

$$= \frac{1}{\sqrt{\mathcal{V}}} \sum_{\mathbf{p},\mathbf{q}} \hat{a}_{\mathbf{p}} e^{i\mathbf{p}\cdot\mathbf{y}} |\mathbf{q}\rangle \langle \mathbf{q}|\mathbf{x}\rangle \quad (12)$$

$$= \frac{1}{\sqrt{\mathcal{V}}} \sum_{\mathbf{p},\mathbf{q}} \hat{a}_{\mathbf{p}} e^{i\mathbf{p}\cdot\mathbf{y}} |\mathbf{q}\rangle \left[\frac{1}{\sqrt{\mathcal{V}}} e^{-i\mathbf{q}\cdot\mathbf{x}} \right] \quad (13)$$

$$= \frac{1}{\mathcal{V}} \sum_{\mathbf{p},\mathbf{q}} e^{i(\mathbf{p}\cdot\mathbf{y}-\mathbf{q}\cdot\mathbf{x})} \hat{a}_{\mathbf{p}} |\mathbf{q}\rangle \quad (14)$$

$$= \frac{1}{\mathcal{V}} \sum_{\mathbf{p},\mathbf{q}} e^{i(\mathbf{p}\cdot\mathbf{y}-\mathbf{q}\cdot\mathbf{x})} \delta_{\mathbf{p}\mathbf{q}} |0\rangle \quad (15)$$

$$= \frac{1}{\mathcal{V}} \sum_{\mathbf{p}} e^{i\mathbf{p}\cdot(\mathbf{y}-\mathbf{x})} |0\rangle \quad (16)$$

$$= \delta^{(3)}(\mathbf{y}-\mathbf{x}) |0\rangle \quad (17)$$

where in the fourth line the momentum annihilation operator $\hat{a}_{\mathbf{p}}$ operating on $|\mathbf{q}\rangle$ produces the vacuum state only if $\mathbf{p} = \mathbf{q}$; otherwise it produces zero.

Thus $\hat{\psi}(\mathbf{y})|\mathbf{x}\rangle$ produces the vacuum state if $\mathbf{y} = \mathbf{x}$ and zero otherwise.