FIELD OPERATORS FOR THE INFINITE SQUARE WELL

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog.

Reference: Tom Lancaster and Stephen J. Blundell, *Quantum Field Theory for the Gifted Amateur*, (Oxford University Press, 2014), Section 4.1.

The creation and annihilation operators $\hat{a}_{\mathbf{p}}^{\dagger}$ and $\hat{a}_{\mathbf{p}}$ create and annihilate a particle in a specific momentum state so, because of the uncertainty principle, such states are completely unlocalized in position. We can construct analogous operators that create and annihilate a particle at a specific position. Such operators are called *field operators*. As you would expect, a field operator, operating at a precise position, must be completely unlocalized in momentum.

The creation and annihilation field operators for a particle in a 3-d infinite square well are defined as

$$\hat{\psi}^{\dagger}(\mathbf{x}) \equiv \frac{1}{\sqrt{\mathcal{V}}}\sum_{\mathbf{p}}\hat{a}_{\mathbf{p}}^{\dagger}e^{-i\mathbf{p}\cdot\mathbf{x}} \tag{1}$$

$$\hat{\psi}(\mathbf{x}) \equiv \frac{1}{\sqrt{\mathcal{V}}}\sum_{\mathbf{p}}\hat{a}_{\mathbf{p}}e^{i\mathbf{p}\cdot\mathbf{x}} \tag{2}$$

where $\mathcal{V}$ is the volume of the square well.

To see that they actually do create and annihilate a particle at position $\mathbf{x}$ we'll have a look at their effect when operating on particular states.

First, we'll apply $\hat{\psi}^{\dagger}(x)$ to the vacuum state.

$$\hat{\psi}^{\dagger}(\mathbf{x})|0\rangle = \frac{1}{\sqrt{\mathcal{V}}}\sum_{\mathbf{p}}e^{-i\mathbf{p}\cdot\mathbf{x}}\hat{a}_{\mathbf{p}}^{\dagger}|0\rangle \tag{3}$$

$$= \frac{1}{\sqrt{\mathcal{V}}}\sum_{\mathbf{p}}e^{-i\mathbf{p}\cdot\mathbf{x}}|\mathbf{p}\rangle \tag{4}$$

We can now insert the unit operator in the form

$$1=\int d^{3}\mathbf{y}\,|\mathbf{y}\rangle\langle\mathbf{y}| \tag{5}$$

and we get

$$\hat{\psi}^{\dagger}(\mathbf{x})|0\rangle = \frac{1}{\sqrt{\mathscr{V}}}\int d^3\mathbf{y}\sum_{\mathbf{p}} e^{-i\mathbf{p}\cdot\mathbf{x}}|\mathbf{y}\rangle\langle\mathbf{y}|\mathbf{p}\rangle \tag{6}$$

$$= \frac{1}{\sqrt{\mathscr{V}}}\int d^3\mathbf{y}\sum_{\mathbf{p}} e^{-i\mathbf{p}\cdot\mathbf{x}}|\mathbf{y}\rangle\left[\frac{1}{\sqrt{\mathscr{V}}}e^{i\mathbf{p}\cdot\mathbf{y}}\right] \tag{7}$$

$$= \int d^3\mathbf{y}\left[\frac{1}{\mathscr{V}}\sum_{\mathbf{p}} e^{i\mathbf{p}\cdot(\mathbf{y}-\mathbf{x})}\right]|\mathbf{y}\rangle \tag{8}$$

$$= \int d^3\mathbf{y}\,\delta^{(3)}(\mathbf{y}-\mathbf{x})|\mathbf{y}\rangle \tag{9}$$

$$= |\mathbf{x}\rangle \tag{10}$$

So the creation operator $\hat{\psi}^{\dagger}(\mathbf{x})$ operating on the vacuum state creates a single particle at position $\mathbf{x}$.

We can now try the annihilation operator $\hat{\psi}(\mathbf{y})$ operating on a one-particle state $|\mathbf{x}\rangle$. We get

$$\hat{\psi}(\mathbf{y})|\mathbf{x}\rangle = \frac{1}{\sqrt{\mathscr{V}}}\sum_{\mathbf{p}}\hat{a}_{\mathbf{p}}e^{i\mathbf{p}\cdot\mathbf{y}}|\mathbf{x}\rangle \tag{11}$$

$$= \frac{1}{\sqrt{\mathscr{V}}}\sum_{\mathbf{p}.\mathbf{q}}\hat{a}_{\mathbf{p}}e^{i\mathbf{p}\cdot\mathbf{y}}|\mathbf{q}\rangle\langle\mathbf{q}|\mathbf{x}\rangle \tag{12}$$

$$= \frac{1}{\sqrt{\mathscr{V}}}\sum_{\mathbf{p}.\mathbf{q}}\hat{a}_{\mathbf{p}}e^{i\mathbf{p}\cdot\mathbf{y}}|\mathbf{q}\rangle\left[\frac{1}{\sqrt{\mathscr{V}}}e^{-i\mathbf{q}\cdot\mathbf{x}}\right] \tag{13}$$

$$= \frac{1}{\mathscr{V}}\sum_{\mathbf{p},\mathbf{q}} e^{i(\mathbf{p}\cdot\mathbf{y}-\mathbf{q}\cdot\mathbf{x})}\hat{a}_{\mathbf{p}}|\mathbf{q}\rangle \tag{14}$$

$$= \frac{1}{\mathscr{V}}\sum_{\mathbf{p},\mathbf{q}} e^{i(\mathbf{p}\cdot\mathbf{y}-\mathbf{q}\cdot\mathbf{x})}\delta_{\mathbf{pq}}|0\rangle \tag{15}$$

$$= \frac{1}{\mathscr{V}}\sum_{\mathbf{p}} e^{i\mathbf{p}\cdot(\mathbf{y}-\mathbf{x})}|0\rangle \tag{16}$$

$$= \delta^{(3)}(\mathbf{y}-\mathbf{x})|0\rangle \tag{17}$$

where in the fourth line the momentum annihilation operator $\hat{a}_{\mathbf{p}}$ operating on $|\mathbf{q}\rangle$ produces the vacuum state only if $\mathbf{p}=\mathbf{q}$; otherwise it produces zero.

Thus $\hat{\psi}(\mathbf{y})|\mathbf{x}\rangle$ produces the vacuum state if $\mathbf{y}=\mathbf{x}$ and zero otherwise.