HOW CAN A FIELD AND ITS CONJUGATE BE INDEPENDENT?

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Reference: Tom Lancaster and Stephen J. Blundell, *Quantum Field Theory for the Gifted Amateur*, (Oxford University Press, 2014), Section 12.1.

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The Lagrangian for a complex scalar field is

$$\mathcal{L}=(\partial_\mu\phi)^\dagger(\partial^\mu\phi)-m^2\phi^\dagger\phi-V\left(\phi^\dagger\phi\right) \tag{1}$$

The usual way of deriving the equations of motion from this Lagrangian is to take ϕ and its conjugate $\phi^\dagger$ to be independent fields. This seems absurd, since if we know ϕ we therefore also know $\phi^\dagger$ as it is just the complex conjugate of ϕ. In their section 12.1, L&B attempt to justify this by saying that we originally formed ϕ from two independent *real* fields ϕ_1 and ϕ_2, so that

L&B call the complex field ψ but to avoid confusing this with the Dirac field, I'll just use ϕ.

$$\phi=\frac{1}{\sqrt{2}}(\phi_1+i\phi_2) \tag{2}$$

$$\phi^\dagger=\frac{1}{\sqrt{2}}(\phi_1-i\phi_2) \tag{3}$$

These equations can be inverted to give

$$\phi_1=\frac{1}{\sqrt{2}}\left(\phi+\phi^\dagger\right) \tag{4}$$

$$\phi_2=-\frac{i}{\sqrt{2}}\left(\phi-\phi^\dagger\right) \tag{5}$$

They then say that this amounts to a change of basis, and that we can vary ϕ_1 alone by setting $\delta\phi=\delta\phi^\dagger$, or ϕ_2 alone by setting $\delta\phi=-\delta\phi^\dagger$. While this is true, as we can see from 3, since setting $\delta\phi=\delta\phi^\dagger$ can be done by setting $\delta\phi_2=0$ and retaining a non-zero $\delta\phi_1$ (and similarly for $\delta\phi=-\delta\phi^\dagger$), it doesn't appear to work in reverse, since there's no obvious way of, for example, setting $\delta\phi=2\delta\phi^\dagger$ as we must be able to do if ϕ and $\phi^\dagger$ are truly independent. If we try this, we get

$$\delta(\phi_1+i\phi_2)=2\delta(\phi_1-i\phi_2) \tag{6}$$

Since both ϕ_1 and ϕ_2 are real, the only way we can satisfy this equation is for the real and imaginary parts to be equal separately, which leads to them both being zero.

I think what is really meant by saying that we can treat ϕ and $\phi^\dagger$ as independent is that we take each of them to define a 'direction' in the space they span, in much the same way that we take x and y to define directions in the complex plane, and the x and y axes form a basis for this plane. We can then express any complex number in the form $a+ib$ where a is understood to be the x (real) component and b is the y (imaginary) component. We can then define a different basis by taking the two directions defined by $a+ib$ and $a-ib$. An arbitrary complex number such as $c+id$ (as expressed in the original x,y basis) can be written in terms of the new basis by solving

$$c+id=\alpha\left(a+ib\right)+\beta\left(a-ib\right) \tag{7}$$

which can be separated into real and imaginary parts to give

$$\alpha+\beta=\frac{c}{a} \tag{8}$$

$$\alpha-\beta=\frac{d}{b} \tag{9}$$

which can then be solved to give

$$\alpha=\frac{1}{2}\left(\frac{c}{a}+\frac{d}{b}\right) \tag{10}$$

$$\beta=\frac{1}{2}\left(\frac{c}{a}-\frac{d}{b}\right) \tag{11}$$

We can vary our position from $c+id$ either by varying along the $a+ib$ direction (by varying α) or along the $a-ib$ direction (by varying β), or by some combination of the two. In other words, we don't vary the actual vectors used to define the direction in the plane, but rather we vary the amount we travel along each of these directions.

In a similar way, we can regard ϕ and $\phi^\dagger$ as defining 'directions' in the space used to construct the field functions, and we then vary the fields by varying how much each of these two quantities contributes to the overall Lagrangian.