MASSIVE ELECTROMAGNETISM: HAMILTONIAN DENSITY

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Reference: Tom Lancaster and Stephen J. Blundell, *Quantum Field Theory for the Gifted Amateur*, (Oxford University Press, 2014), Section 13.2.

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In their section 13.2, L&B introduce the Lagrangian for a version of electromagnetism in which the photon has a non-zero mass. This is

$$\mathcal{L}=-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}+\frac{1}{2}m^{2}A_{\mu}A^{\mu} \tag{1}$$

The first term is the usual term for a free electromagnetic field, and the second term introduces the mass of the photon.

The quantities $\Pi^{\mu\nu}$ now become a second-rank tensor, which is calculated from the usual formula

$$\Pi^{\mu\nu}(x)=\frac{\partial\mathcal{L}}{\partial(\partial_{\mu}A_{\nu})} \tag{2}$$

$$=\partial^{\nu}A^{\mu}-\partial^{\mu}A^{\nu}=-F^{\mu\nu} \tag{3}$$

The momentum densities that enter into the Hamiltonian are

$$\Pi^{0i}=\partial^{i}A^{0}-\partial^{0}A^{i} \tag{4}$$

$$\Pi^{00}=0 \tag{5}$$

From the original definition of the electromagnetic field tensor and the relation 3

$$F^{\mu\nu}=\begin{bmatrix}0 & -E_x & -E_y & -E_z\\ E_x & 0 & -B_z & B_y\\ E_y & B_z & 0 & -B_x\\ E_z & -B_y & B_x & 0\end{bmatrix} \tag{6}$$

$$F_{\mu\nu}=\begin{bmatrix}0 & E_x & E_y & E_z\\ -E_x & 0 & -B_z & B_y\\ -E_y & B_z & 0 & -B_x\\ -E_z & -B_y & B_x & 0\end{bmatrix} \tag{7}$$

we see that we can write

$$\Pi^{0i} = -F^{0i} = E^i \tag{8}$$

Since we're not talking about 'real' electromagnetism, this use of the term E^i does not refer to a real electric field; rather it's just a notational convenience.

From here, L&B give a sketchy derivation of the Hamiltonian from the Lagrangian, but I think it merits a more detailed explanation. First we note that

$$-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} = \frac{1}{2}\left(\mathbf{E}^2 - \mathbf{B}^2\right) \tag{9}$$

which is obtained by substituting for $F_{\mu\nu}$ directly from 6. Thus the Lagrangian becomes

$$\mathcal{L} = \frac{1}{2}\left(\mathbf{E}^2 - \mathbf{B}^2\right) + \frac{1}{2}m^2 A_\mu A^\mu \tag{10}$$

The Hamiltonian is defined by

$$\mathcal{H} = \Pi^{0\nu}\dot{A}_\mu - \mathcal{L} \tag{11}$$

From 4, 8 and 10 we have

$$\mathcal{H} = E^i \dot{A}_i - \frac{1}{2}\left(\mathbf{E}^2 - \mathbf{B}^2\right) - \frac{1}{2}m^2 A_\mu A^\mu \tag{12}$$

The 3-vector $\mathbf{A}$ has components A^i (with an upper index), which is given by

$$A^i = -A_i \tag{13}$$

so we can write

$$\mathcal{H} = -\mathbf{E}\cdot\dot{\mathbf{A}} - \frac{1}{2}\left(\mathbf{E}^2 - \mathbf{B}^2\right) - \frac{1}{2}m^2\left(\left(A^0\right)^2 - \mathbf{A}^2\right) \tag{14}$$

This is result I in L&B's side-note 13 on p. 121.

Now we use the fact that the components of the gradient of A^0 are given by

$$\left(\nabla A^0\right)_i = \partial_i A^0 = \frac{\partial A^0}{\partial x^i} = -\partial^i A^0 \tag{15}$$

From 4 we have

$$E^i = \partial^i A^0 - \partial^0 A^i \tag{16}$$

$$\mathbf{E} = -\nabla A^0 - \dot{\mathbf{A}} \tag{17}$$

$$\dot{\mathbf{A}} = -\nabla A^0 - \mathbf{E} \tag{18}$$

We also have the vector identity

$$\mathbf{E}\cdot\nabla A^0=\nabla\cdot\left(\mathbf{E}A^0\right)-A^0\nabla\cdot\mathbf{E} \tag{19}$$

Therefore the first term in 14 is

$$-\mathbf{E}\cdot\dot{\mathbf{A}}=\mathbf{E}^2+\mathbf{E}\cdot\nabla A^0 \tag{20}$$

$$=\mathbf{E}^2+\nabla\cdot\left(\mathbf{E}A^0\right)-A^0\nabla\cdot\mathbf{E} \tag{21}$$

Substituting this into 14 we have

$$\mathcal{H}=\nabla\cdot\left(\mathbf{E}A^0\right)-A^0\nabla\cdot\mathbf{E}+\frac{1}{2}\left(\mathbf{E}^2+\mathbf{B}^2\right)-\frac{1}{2}m^2\left(\left(A^0\right)^2-\mathbf{A}^2\right) \tag{22}$$

We can now eliminate A^0. The equations of motion, derived by applying the Euler-Lagrange equations to 1 are given by L&B's equation 13.18:

$$\partial_\mu F^{\mu\nu}+m^2A^\nu=0 \tag{23}$$

The $\nu=0$ equation from this set gives us, from 6:

$$A^0=-\frac{1}{m^2}\partial_\mu F^{\mu 0} \tag{24}$$

$$=-\frac{1}{m^2}\partial_\mu E^\mu \tag{25}$$

$$=-\frac{1}{m^2}\nabla\cdot\mathbf{E} \tag{26}$$

Substituting into 22 we have

$$\mathcal{H}=\nabla\cdot\left(\mathbf{E}A^0\right)+\frac{1}{m^2}\left(\nabla\cdot\mathbf{E}\right)^2+\frac{1}{2}\left(\mathbf{E}^2+\mathbf{B}^2\right)-\frac{1}{2m^2}\left(\nabla\cdot\mathbf{E}\right)^2+\frac{m^2}{2}\mathbf{A}^2 \tag{27}$$

$$=\nabla\cdot\left(\mathbf{E}A^0\right)+\frac{1}{2m^2}\left(\nabla\cdot\mathbf{E}\right)^2+\frac{1}{2}\left(\mathbf{E}^2+\mathbf{B}^2\right)+\frac{m^2}{2}\mathbf{A}^2 \tag{28}$$

We now observe that the first term $\nabla\cdot\left(\mathbf{E}A^0\right)$ is a total divergence, and apply the usual argument that when we integrate $\mathcal{H}$ over all space, this gets converted to a surface integral using Gauss's theorem, and all fields are assumed to vanish at infinity so this term contributes nothing, and can be dropped. The final result is then

$$\mathcal{H}=\frac{1}{2m^2}\left(\nabla\cdot\mathbf{E}\right)^2+\frac{1}{2}\left(\mathbf{E}^2+\mathbf{B}^2\right)+\frac{m^2}{2}\mathbf{A}^2 \tag{29}$$

A final note is that it was only possible to eliminate A^0 using 26 because $m \neq 0$. Thus we *cannot* obtain a theory of 'proper' electrodynamics (where the photon is massless) simply by setting $m = 0$ in the above derivation. This is also obvious from the final result, since the first term involves a $\frac{1}{m^2}$ factor which blows up for $m \to 0$.

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