ALTERNATIVE PROOF OF A TRIG IDENTITY

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog and include the title or URL of this post in your comment.

Post date: 25 December 2024.

The trig identity

$$\sin^2 x + \cos^2 x = 1 \tag{1}$$

is usually derived for real variables using a right-angled triangle with unit hypotenuse. When we generalize trig functions to the complex domain, we can find a proof that this identity still holds for complex variables.

Theorem 1. *We propose*

$$\sin^2 z + \cos^2 z = 1 \tag{2}$$

where z is any complex number.

Proof. First, we observe that both $\sin z$ and $\cos z$ are entire functions. This follows from their definitions and the fact that the exponential function is entire.

$$\begin{aligned} \sin z &\equiv \frac{e^{iz} - e^{-iz}}{2i} \\ \cos z &\equiv \frac{e^{iz} + e^{-iz}}{2} \end{aligned} \tag{3}$$

The derivatives follow:

$$\frac{d}{dz}\sin z = \frac{ie^{iz} + ie^{-iz}}{2i} = \cos z \tag{4}$$

$$\frac{d}{dz}\cos z = \frac{ie^{iz} - ie^{-iz}}{2} = -\frac{e^{iz} - e^{-iz}}{2i} = -\sin z \tag{5}$$

Now let

$$f(z) = \sin^2 z + \cos^2 z \tag{6}$$

Then

$$f'(z) = 2\sin z \cos z - 2\cos z \sin z = 0 \tag{7}$$

This is true for all z. Therefore $f(z)$ is a constant function over the entire complex plane.

The value at $z = 0$ is

$$f(0) = \sin^2 0 + \cos^2 0 = 0 + 1 = 1 \tag{8}$$

Therefore $\sin^2 z + \cos^2 z = 1$ over the entire complex plane. □