# ANALYTIC AND ENTIRE FUNCTIONS

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A complex function $f(z)$ is said to be *analytic* on an open set $G$ if it has a derivative at every point of $G$. Although there are some pathological cases where a function has a derivative at only a single point or along a single curve, these cases rarely, if at all, occur in physics. In these pathological cases, the function could well be analytic over the open set that doesn't include such points or curves. We won't concern ourselves with such functions.

analytic function

An *entire* function is a function that is analytic over the entire complex plane. All polynomials are entire functions. We've seen that the Cauchy-Riemann equations (CR) are a necessary condition for a function to be differentiable at a point $z_0$ but we noted that this is not necessarily sufficient. The precise statement is in the following theorem.

entire function

**Theorem 1.** *For a function $f(z) = u(x,y) + iv(x,y)$ that is defined in an open set $G$ containing the point $z_0$, $f(z)$ is differentiable at $z_0$ if the first partial derivatives of $u$ and $v$ exist, satisfy the Cauchy-Riemann equations and are continuous at $z_0$.*

The proof of this theorem is rather involved and doesn't add much to our understanding of its application, so we won't go into it here. Interested readers can find the proof in Section 2.4 of Saff and Snider's book.

We'll give a few examples of functions that are analytic or otherwise.

**Example 1.** $f(z) = 8\overline{z} + i$. Since $\overline{z}$ is not differentiable anywhere as the CR equations are not satisfied, this function is not analytic.

**Example 2.** $f(z) = \frac{z}{\overline{z}+2}$. At first glance, this would appear not to be analytic because of the $\overline{z}$ in the denominator. If we rationalize the function we get (using Maple):

$$u(x,y) = \frac{x^2 - y^2 + 2x}{x^2 + y^2 + 4x + 4} \tag{1}$$

$$v(x,y) = \frac{2y(x+1)}{x^2 + y^2 + 4x + 4} \tag{2}$$

From here, we can grind through the algebra (again using Maple) of calculating the CR equations and find that, as expected, they are not satisfied so this function is not analytic. I'm not sure if there is a clever, quick way of doing this though. Comments welcome.

**Example 3.** $f(z)=z+\frac{\overline{z}}{|z|^2}$. Here, we might expect the function not to be analytic as it contains both $\overline{z}$ and $|z|$, neither of which is differentiable. However, in this case, looks can be deceiving. The second term is equivalent to

$$\frac{\overline{z}}{|z|^2}=\frac{\overline{z}}{z\overline{z}}=\frac{1}{z} \tag{3}$$

so is actually a rational function of polynomials, which we know to be differentiable except at $z=0$. (Thanks to an anonymous commenter for pointing this out.) However, it's a useful exercise to show that the CR equations are valid in this case.

If we write out the real and imaginary parts, we get

$$f(z)=x+\frac{x}{x^2+y^2}+i\left(y-\frac{y}{x^2+y^2}\right) \tag{4}$$

$$u(x,y)=x+\frac{x}{x^2+y^2} \tag{5}$$

$$v(x,y)=y-\frac{y}{x^2+y^2} \tag{6}$$

Calculating CR, we get (use Maple, or the quotient rule if you want to do it by hand):

$$\frac{\partial u}{\partial x}=1+\frac{y^2-x^2}{(x^2+y^2)^2} \tag{7}$$

$$\frac{\partial v}{\partial y}=1+\frac{y^2-x^2}{(x^2+y^2)^2} \tag{8}$$

Thus $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}$. For the other two CR equations we have

$$\frac{\partial u}{\partial y}=\frac{-2xy}{(x^2+y^2)^2} \tag{9}$$

$$\frac{\partial v}{\partial x}=\frac{2xy}{(x^2+y^2)^2} \tag{10}$$

Thus $\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$ so the other two CR equations are also satisfied. The derivatives exist and are continuous everywhere except $z=0$, so this function is actually analytic, although it is not entire, since it excludes $z=0$.

**Example 4.** $f(z)=x^2+y^2+y-2+ix$. In this case

$$u=x^2+y^2+y-2 \tag{11}$$
$$v=x \tag{12}$$

We have

$$\frac{\partial u}{\partial x}=2x \tag{13}$$
$$\frac{\partial v}{\partial y}=0\neq\frac{\partial u}{\partial x} \tag{14}$$
$$\frac{\partial u}{\partial y}=2y+1 \tag{15}$$
$$\frac{\partial v}{\partial x}=1\neq-\frac{\partial u}{\partial y} \tag{16}$$

so this is not analytic anywhere.

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