ANALYTIC BRANCHES OF ROOTS

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If we want to find a branch of a function involving roots that is analytic in a certain domain, we can use the principal branch of the logarithm Log z to define such a branch. The principal branch of the logarithm has a branch cut along the negative real axis, so we need to find a branch of the given function that avoids this region in the logarithm.

Example 1. Find a branch of $\left(z^2-1\right)^{1/2}$ that is analytic in the unit disk $|z|<1$.

The representation in terms of the logarithm is

$$\left(z^2-1\right)^{1/2}=e^{(1/2)\text{Log}\left(z^2-1\right)} \tag{1}$$

However, the principal branch of the logarithm has a branch cut along the negative real axis, so this representation won't work, as the argument of the logarithm is negative and real when $z=x\in(-1,1)$ or $z=iy$ for any value of y. That is, it has branch cuts within the unit circle. We can rewrite it as

$$\left(z^2-1\right)^{1/2}=\left[-\left(1-z^2\right)\right]^{1/2} \tag{2}$$

$$=i\left(1-z^2\right)^{1/2} \tag{3}$$

where we've chosen $(-1)^{1/2}=i$ in the last line.

The branch cuts now appear only when $|z|>1$, so there are no branch cuts inside the unit disk. We therefore have

$$\left(z^2-1\right)^{1/2}=ie^{(1/2)\text{Log}\left(1-z^2\right)} \tag{4}$$

Example 2. Find a branch of $\left(4+z^2\right)^{1/2}$ that is analytic in the complex plane slit along the line from $-2i$ to $2i$.

The branch cuts of $\left(4+z^2\right)^{1/2}$ occur when $4+z^2$ is real and negative, which occurs in the intervals $y\in(-\infty,-2i)$ and $y\in(2i,\infty)$. Thus the function as it stands has branch cuts in the complement of the desired interval. We can try

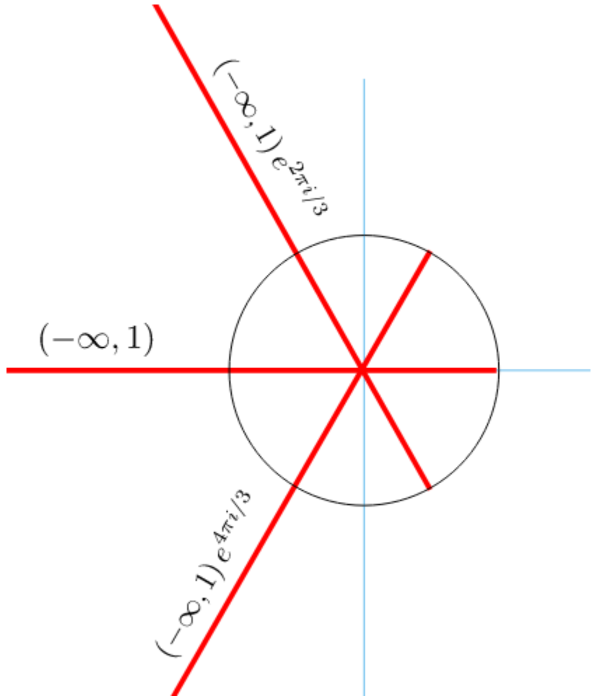

FIGURE 1. Branch cuts of $\left(z^3-1\right)^{1/3}$.

$$\left(4+z^2\right)^{1/2}=z\left(1+\frac{4}{z^2}\right)^{1/2} \tag{5}$$

The square root is of a negative real quantity if z lies in the interval $(-2i,2i)$, so the desired analytic branch is

$$\left(4+z^2\right)^{1/2}=ze^{(1/2)\mathrm{Log}\,\left(1+4/z^2\right)} \tag{6}$$

Example 3. Find a branch of $\left(z^4-1\right)^{1/2}$ that is analytic outside the unit circle $|z|>1$.

The branch points occur when $z^4-1=0$, which are at $z=\pm1,\pm i$. The function z^4-1 is negative and real on the intervals $(-1,1)$ and $(-i,i)$ (since $i^4=(-i)^4=1$). Thus all the branch cuts occur only within the unit circle, so the function is analytic as it stands. That is, we can take

$$\left(z^4-1\right)^{1/2}=e^{(1/2)\mathrm{Log}\,\left(z^4-1\right)} \tag{7}$$

Example 4. Find a branch of $\left(z^3-1\right)^{1/3}$ that is analytic outside the unit circle $|z|>1$.

The branch points occur at $z = 1, e^{2\pi i/3}, e^{4\pi i/3}$. One branch cut is the interval $(-\infty, 1)$. The other 2 branch cuts correspond to multiplying this interval by the other two branch points. That is, there are branch cuts along the lines $(-\infty, 1)\, e^{2\pi i/3}$ and $(-\infty, 1)\, e^{4\pi i/3}$ (see Fig. 1). If we try

$$\left(z^3 - 1\right)^{1/3} = z\left(1 - \frac{1}{z^3}\right)^{1/3} \tag{8}$$

then the branch cuts of $\left(1 - \frac{1}{z^3}\right)^{1/3}$ occurs only when $\left|z^3\right| < 1$, that is when z is inside the unit circle, so we can take

$$\left(z^3 - 1\right)^{1/3} = z e^{(1/3)\operatorname{Log}\left(1 - 1/z^3\right)} \tag{9}$$