ANALYTIC CONTINUATION

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog and include the title or URL of this post in your comment.

Post date: 13 November 2021.

In Coleman's discussion of Mandelstam variables, he makes use of a concept from complex variable theory called *analytic continuation*. As this may not be familiar to those with a basic knowledge of physics, I thought I'd give an explanation here.

My discussion will be oriented towards physicists, so I won't be doing any rigorous proofs. Such proofs can be found in most textbooks on complex variable theory.

First, we'll need a few definitions. A function $f(z)$ of a complex variable z is *analytic* over a domain if it is 'well-behaved' over that domain. Basically this means that the function doesn't have any poles (singularities, in physics-speak), and is differentiable to all orders. The function $f(z)=z^2$ is analytic over the entire complex plane, while $f(z)=\frac{1}{1-z}$ is analytic everywhere except at $z=1$, where it has a pole.

A function that is analytic everywhere except for a few isolated points is called *meromorphic*. The function $f(z)=\frac{1}{1-z}$ is meromorphic.

A result from complex variable theory that we will need is the theorem:

Theorem 1. *A meromorphic function can be represented by a Taylor series expansion about every point where it is analytic.*

As a reminder, the general form for a Taylor series is

$$f(z)=\sum_{n=0}^{\infty}\frac{f^{(n)}(z_0)}{n!}(z-z_0)^n \tag{1}$$

where $f^{(n)}(z_0)$ is the nth derivative of f evaluated at $z=z_0$.

For $f(z)=z^2$, we have a finite series that just reduces back to $f(z)=z^2$:

$$\begin{aligned} f(z) &= z_0^2+\frac{2z_0}{1!}(z-z_0)+\frac{2}{2!}(z-z_0)^2 && (2)\\ &= z_0^2+2z_0z-2z_0^2+z^2-2z_0z+z_0^2 && (3)\\ &= z^2 && (4)\end{aligned}$$

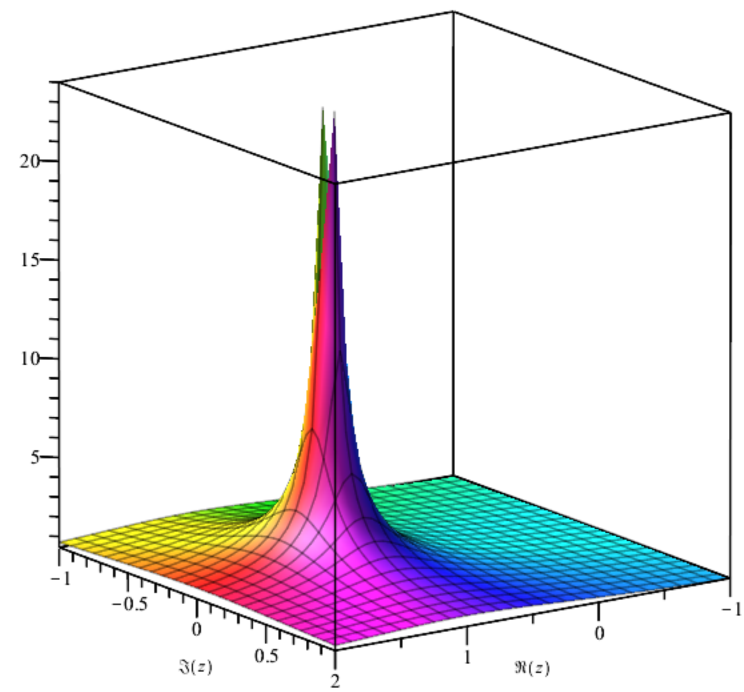

FIGURE 1. A plot of $\left|\frac{1}{1-z}\right|$ in the complex plane.

This expansion is valid for any value of z, since $f(z)$ is analytic over the entire complex plane.

The function $f(z) = \frac{1}{1-z}$ has the series expansion (as you probably remember from a high school math class):

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \tag{5}$$

A plot of $\left|\frac{1}{1-z}\right|$ in the complex plane (Fig. 1) shows the pole at $z = 1$.

This can be derived by starting with a finite series:

$$S_N = \sum_{n=0}^{N} z^n = 1 + z + \ldots + z^N \tag{6}$$

We multiply both sides by z to get

$$zS_N = z + z^2 + \ldots + z^N + z^{N+1} \tag{7}$$

Subtracting the second equation from the first we get

$$(1-z)\,S_N = 1 - z^{N+1} \tag{8}$$

and then dividing through by $1-z$ we obtain

$$S_N = \frac{1-z^{N+1}}{1-z} \tag{9}$$

We can take the limit $N \to \infty$ which results in $z^{N+1} \to 0$, but *only if* $|z| < 1$. Thus 5 converges only if $|z| < 1$.

In fact, 5 is the Taylor expansion of $\frac{1}{1-z}$ about the point $z = 0$, so in this case we see that the Taylor expansion of $f(z)$ about this point is valid only within the circle $|z| < 1$. However, although the function $\frac{1}{1-z}$ has a pole at $z = 1$, it seems perfectly well behaved everywhere else in the complex plane, so can we find a series expansion that converges in some or all of the parts of the complex plane with $|z| > 1$?

The trick to finding such an extension is to realize that, within the valid region of $|z| < 1$, we can write a Taylor expansion about any other point. Since we know the exact form of $f(z) = \frac{1}{1-z}$, we can calculate the value of the function and all its derivatives at any point, so we can (in principle) write out the Taylor series to any degree of accuracy we want.

For anything beyond the first few terms the calculation gets quite tedious, so it's best to use Maple to do the work. First, we'll look at the domain in which the series 5 converges (Fig. 2).

Now we can choose a point within this circle (say $z = 0.9i$) and write out the Taylor series about this point. The calculation is quite messy, but the first few terms (with 3 decimal point accuracy) are

$$f(z) = \frac{1}{1-z} = \sum_{n=0}^{\infty} \frac{f^{(n)}(0.9i)}{n!}(z-0.9i)^n \tag{10}$$

$$= (0.552+0.497i) + (0.058+0.549i)(z-0.9i) + (-0.241+0.332i)(z-0.9i)^2 + \ldots \tag{11}$$

$$\tag{12}$$

I actually worked out the expansion out to the 50th term, so the results should be fairly accurate, assuming that the series converges.

As a check, we can work out the value at a few points with both expansions (Table 1).

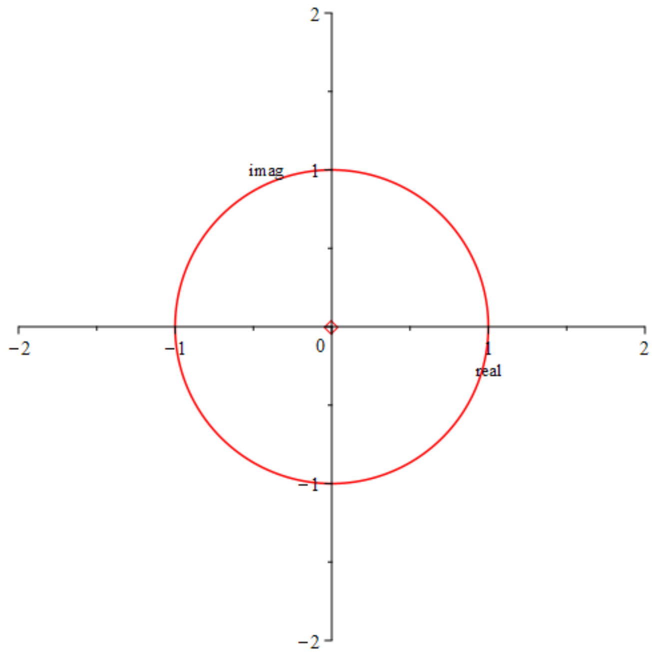

FIGURE 2. Region of convergence of the series 5.

z	Eqn 5	Eqn 10	$\frac{1}{1-z}$
0	1	1	1
0.5	2	2	2
$0.5i$	$0.8+0.4i$	$0.8+0.4i$	$0.8+0.4i$
i	$1+i$	$0.5+0.5i$	$0.5+0.5i$
$2i$	$2.25\times10^{14}+4.5\times10^{14}i$	$0.2+0.4i$	$0.2+0.4i$
2	1.13×10^{15}	$-4\times10^{10}+8.52\times10^{9}i$	-1

TABLE 1. Comparison of exact value of $\frac{1}{1-z}$ and two Taylor series.

We see that for points within the domain of convergence for the first series 5, the results of both series are correct. When we try the point $z=i$, however, things go wrong with 5, because this point is on the boundary of the domain of convergence. This domain is an open set (meaning it doesn't include its boundary), so we can expect things to go wrong here.

If we go even further afield on the imaginary axis with $z=2i$, we see that 5 now diverges, but 10 still gives the correct answer. Going out to $z=2$ on the real axis, however, causes both series to diverge.

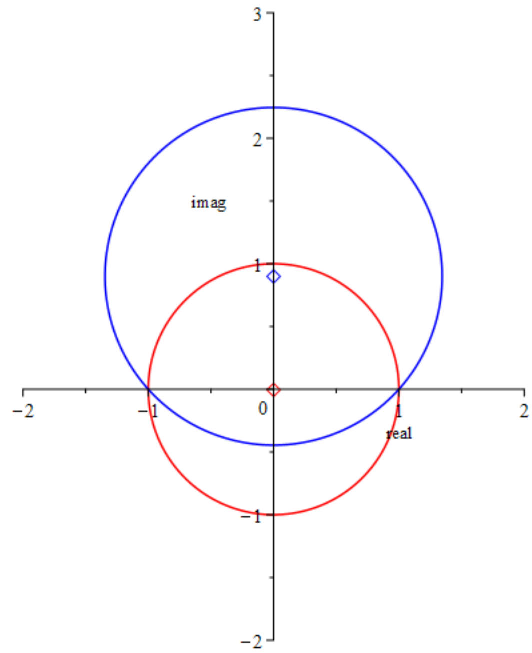

FIGURE 3. Analytic continuation of $\frac{1}{1-z}$. The coloured diamonds give the centres of the respective circles.

What is happening is that the expansion 10 around the point $z = 0.9i$ has a new domain of convergence. This domain can be found by drawing the largest circle centred at $z = (0, 0.9i)$ that doesn't include the pole at $z = 1$. In Fig. 3, this domain is drawn as the blue circle.

All points in the intersection of the blue and red circles will give the correct Taylor series, but for those regions within the blue circle but outside the red circle, the original series 5 will not converge to the correct value. We can see that the point $z = 2i$ is in this latter region, so the series 10 gives the correct result while 5 does not. The point $z = 2$ is outside both domains of convergence, so gives the wrong result from both series.

This process of expanding the Taylor series about a different point to extend the region of convergence is what is known as *analytic continuation*. We can continue the process by selecting a new point that is somewhere within the blue circle and writing down the Taylor series about that point. The domain of convergence of this new series will be the largest circle centred at the new point that does not include the point $z = 1$. Thus by drawing enough circles, we can analytically continue the function to cover the entire complex plane with the exception of the point $z = 1$.

We can see the effects in 3-d graphs as follows. Fig. 4 shows a plot of the first Taylor series 5 (with 51 terms, from $n = 0$ to $n = 50$). The green

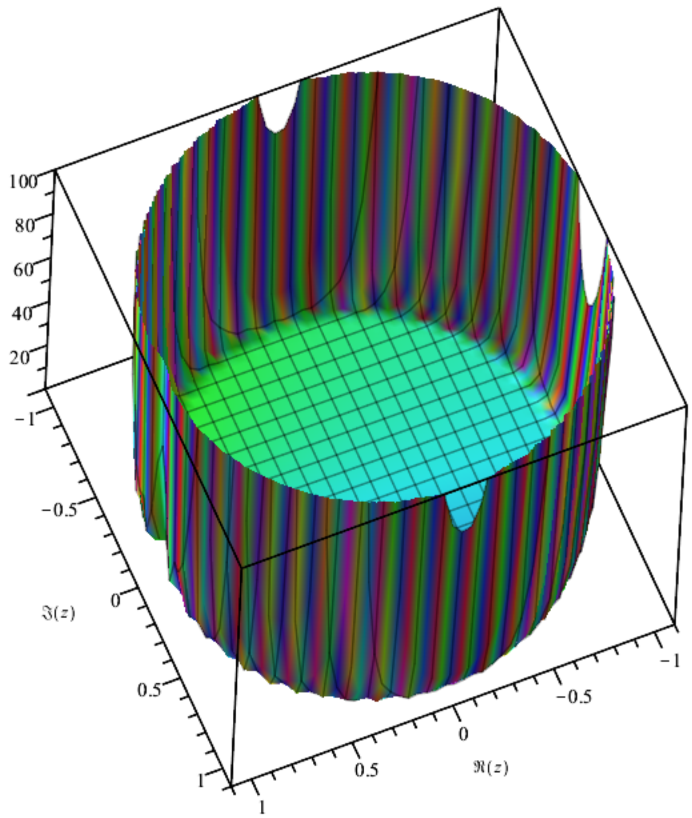

FIGURE 4. Plot of $\sum_{n=0}^{50} z^n$. The green circular region is the domain of convergence.

circular region shows where the series converges, and we see that this is a unit circle centred at the origin. Beyond the boundary of the unit circle, the series diverges rapidly.

We can also plot the second series 10 (Fig. 5). Again, we see a circular domain of convergence, but this is centred at $z_0 = 0.9i$. Again, beyond the circular region of convergence, the series diverges rapidly, but the domain for this series covers a different region than the first series.

PINGBACKS

Pingback: Crossing symmetry

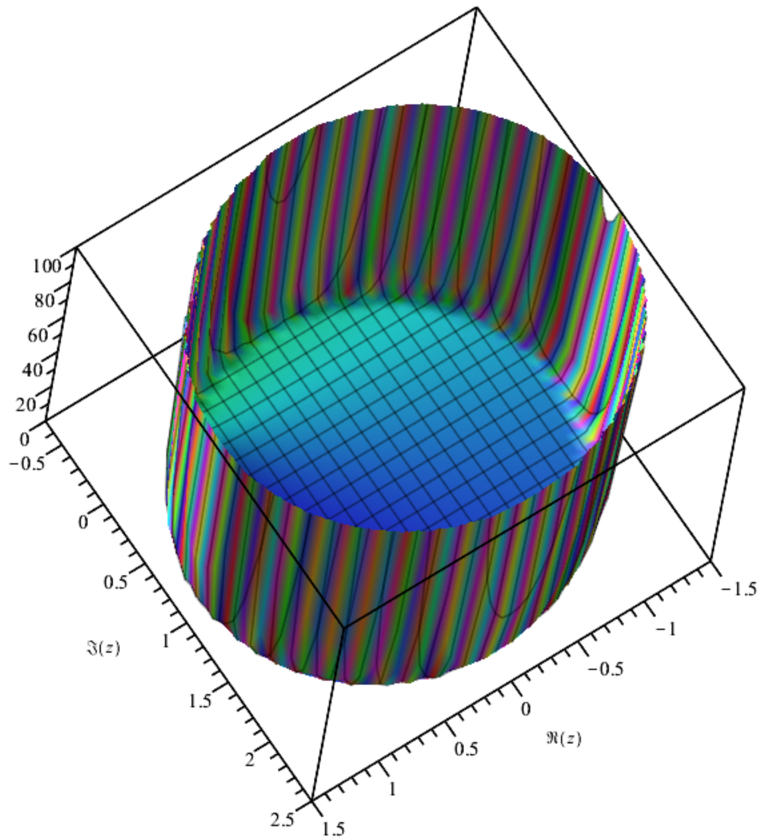

FIGURE 5. Plot of $\sum_{n=0}^{50} \frac{f^{(n)}(0.9i)}{n!}(z-0.9i)^n$. The blue circular region shows the domain of convergence.