ARGUMENT OF A COMPLEX NUMBER

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When a complex number z is written in polar form it has the form

$$z = r(\cos\theta + i\sin\theta) \equiv r\text{cis}\theta \tag{1}$$

where r is the magnitude and θ is the argument, written as $\arg z = \theta$ with a lower-case 'a'. The principal argument, written $\text{Arg}z$, satisfies $-\pi < \text{Arg}z \le \pi$. The value of θ is determined by

$$\theta = \arctan\frac{y}{x} \tag{2}$$

although some adjustment is required if θ lies in the second or third quadrant, since the arctan function is commonly defined to give an angle in the range $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. A formula for finding $\arg z$ is

$$\arg(x+iy) = \begin{cases} \arctan\frac{y}{x} + \frac{\pi}{2}[1-\text{sgn}(x)] & \text{if } x \ne 0 \\ \frac{\pi}{2}\text{sgn}(y) & \text{if } x = 0 \text{ and } y \ne 0 \\ \text{undefined} & \text{if } x = y = 0 \end{cases} \tag{3}$$

To check this, we consider a value in each of the four quadrants. If $x > 0$ and $y > 0$, then $\text{sgn}(x) = 1$ and $\arg(x+iy) = \arctan\frac{y}{x}$, which lies in the range $0 < \arctan\frac{y}{x} \le \frac{\pi}{2}$. If $x < 0$ and $y > 0$, we are in the second quadrant, and $\arg(x+iy) = \arctan\frac{y}{x} + \pi$. In this case, $-\frac{\pi}{2} \le \arctan\frac{y}{x} \le 0$, so adding π to this gives a range $\left[\frac{\pi}{2}, \pi\right]$ which is in the second quadrant. If $x < 0$ and $y < 0$, then $\arctan\frac{y}{x}$ lies in the first quadrant, so again, adding π to this gives a value in the third quadrant. Finally, if $x > 0$ and $y < 0$, we get $\arctan\frac{y}{x}$ in the fourth quadrant, as required.

Note that this formula does *not* give correct values for $\text{Arg}z$ for values of z in the third quadrant, since here $-\pi < \text{Arg}z < -\frac{\pi}{2}$. In this case, we need to subtract 2π from the value of $\arg z$ given by 3.

Example 1. $\text{Arg}(-6-6i)$. This lies in the third quadrant, so we have from 3

$$\arg(-6-6i) = \arctan\frac{-6}{-6} + \pi = \frac{5\pi}{4} \tag{4}$$

To get the principal argument, we need to subtract 2π, so we have

$$\text{Arg}\,(-6-6i) = -\frac{3\pi}{4} \tag{5}$$

Example 2. $\text{Arg}\,(-\pi)$. This lies on the negative x axis, so $\text{Arg}\,(-\pi) = \pi$.

Example 3. $\text{Arg}\,(10i)$. This lies on the positive y axis, so $\text{Arg}\,(10i) = \frac{\pi}{2}$.

Example 4. $\text{Arg}\left(\sqrt{3}-i\right)$. This lies in the fourth quadrant, so

$$\text{Arg}\left(\sqrt{3}-i\right) = \arctan\left(-\frac{1}{\sqrt{3}}\right) = -\frac{\pi}{6} \tag{6}$$

A few statements about the principal argument follow. We can test each statement for validity.

Example 5. First,

$$\text{Arg}\,(z_1 z_2) = \text{Arg}z_1 + \text{Arg}z_2 \tag{7}$$

This is true only if the sum lies in the range $(-\pi,\pi]$. If not, then 2π must be added or subtracted to get the correct principal argument.

Example 6. Second,

$$\text{Arg}\bar{z} = -\text{Arg}z \tag{8}$$

This is true unless z is a negative real number, where $\text{Arg}z = \pi$. In this case, the argument is unchanged. (The argument is also unchanged if z is a positive real, where $\text{Arg}z = 0$.)

Example 7. Third,

$$\text{Arg}\left(\frac{z_1}{z_2}\right) = \text{Arg}z_1 - \text{Arg}z_2 \tag{9}$$

As with the product above, this is true provided the difference on the RHS gives a value in the range $(-\pi,\pi]$. If not, then 2π must be added or subtracted to get the correct principal argument.

Example 8. Fourth,

$$\arg z = \text{Arg}z + 2\pi k \tag{10}$$

where k is an integer (positive or negative). This is always true (unless $z = 0$, in which case the argument is undefined).

Example 9. Finally, we consider the product

$$z = (1+i)(5-i)^4 \tag{11}$$

We can expand the second factor using the binomial theorem (I used Maple to simplify):

$$(5-i)^4 = 476 - 480i \tag{12}$$

The product then becomes

$$z = 956 - 4i \tag{13}$$

Thus

$$\arg z = \arctan\left(-\frac{4}{956}\right) = -\arctan\left(\frac{1}{239}\right) \tag{14}$$

From 11 we have

$$\arg\left[(5-i)^4\right] = -\arctan\left(\frac{1}{239}\right) \tag{15}$$

$$= \arctan\left(\frac{1}{1}\right) + 4\arctan\left(\frac{-1}{5}\right) \tag{16}$$

$$= \frac{\pi}{4} - 4\arctan\left(\frac{1}{5}\right) \tag{17}$$

So we have

$$\frac{\pi}{4} = 4\arctan\left(\frac{1}{5}\right) - \arctan\left(\frac{1}{239}\right) \tag{18}$$

giving a rather unusual expression for π.