ASSOCIATED LEGENDRE FUNCTIONS

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One of the differential equations that turns up in the solution in the three-dimensional Schrödinger equation is Legendre's equation:

$$\frac{d}{dx}\left((1-x^2)\frac{dP}{dx}\right)+\left[l(l+1)-\frac{m^2}{1-x^2}\right]P=0 \tag{1}$$

We saw in another post that if we take $m=0$, the solutions are the Legendre polynomials $P_l(x)$. That is, they are the solutions of

$$\frac{d}{dx}\left((1-x^2)\frac{dP_l}{dx}\right)+l(l+1)P_l = 0 \tag{2}$$

$$(1-x^2)P_l''-2xP_l'+l(l+1)P_l = 0 \tag{3}$$

To find the solution to the general Legendre equation 1 we can, oddly enough, start with the simpler equation. We can use Leibniz's formula to differentiate the $m=0$ equation m times. For each term in the above equation, we get

$$[(1-x^2)P_l'']^{(m)} = (1-x^2)P_l^{(m+2)}-2mxP_l^{(m+1)}-m(m-1)P_l \tag{4}$$

$$[-2xP_l']^{(m)} = -2xP_l^{(m+1)}-2mP_l^{(m)} \tag{5}$$

$$[l(l+1)P_l]^{(m)} = l(l+1)P_l^{(m)} \tag{6}$$

Adding up these three terms, we get

$$(1-x^2)P_l^{(m+2)}-2x(m+1)P_l^{(m+1)}+[l(l+1)-m^2-m]P_l=0 \tag{7}$$

We can simplify the notation a bit by defining $u(x)\equiv P_l^{(m)}$:

$$(1-x^2)u''-2x(m+1)u'+[l(l+1)-m^2-m]u=0 \tag{8}$$

This still doesn't look much like Legendre's general equation, so next we use the substitution (I know, I know - how would you think of this?)

(9)

$$u = v(1-x^2)^{-m/2}$$

(10)

$$u' = v'(1-x^2)^{-m/2} + mxv(1-x^2)^{-(m+2)/2}$$

(11)

$$= \left(v' + \frac{mxv}{(1-x^2)}\right)(1-x^2)^{-m/2}$$

(12)

$$u'' = \left(v'' + \frac{mxv'}{1-x^2} + \frac{mv}{1-x^2} + \frac{2mx^2v}{(1-x^2)^2}\right)(1-x^2)^{-m/2} + \left(v' + \frac{mxv}{1-x^2}\right)xm(1-x^2)^{-(m+2)/2}$$

(13)

$$= \left(v'' + \frac{2mxv'}{1-x^2} + \frac{mv}{1-x^2} + \frac{m(m+2)x^2v}{(1-x^2)^2}\right)(1-x^2)^{-m/2}$$

We can substitute these terms into 8 and collect terms to get, finally, Legendre's equation back again:

(14)

$$(1-x^2)v'' - 2xv' + mv + \frac{v}{1-x^2}(m(m+2)x^2 - 2m(m+1)x^2) + [l(l+1) - m^2 - m]v = 0$$

$$(1-x^2)v'' - 2xv' + mv - \frac{m^2x^2v}{1-x^2} + l(l+1)v - m^2\frac{1-x^2}{1-x^2}v - mv = 0 \tag{15}$$

$$(1-x^2)v'' - 2xv' + l(l+1)v - \frac{m^2}{1-x^2}v = 0 \tag{16}$$

$$\frac{d}{dx}((1-x^2)v') + l(l+1)v - \frac{m^2}{1-x^2}v = 0 \tag{17}$$

That is, the function $v(x)$ is a solution of the general Legendre equation with an arbitrary value of m. These solutions are called associated Legendre functions, and from the definitions above, we get

$$\begin{aligned} v(x) &= u(x)(1-x^2)^{m/2} && (18)\\ &= (1-x^2)^{m/2}\frac{d^mP_l(x)}{dx^m} && (19)\\ &\equiv P_l^m(x) && (20) \end{aligned}$$

where the term $P_l^m(x)$ is the symbol usually reserved for the associated Legendre function with indexes l and m.

Although the derivation is fairly straightforward once it is laid in front of you, it is still a bit of magic when you see the substitution that needs to be made to end up with Legendre's general equation.

From this formula we can use the explicit sum version of the Legendre polynomials to get an explicit formula for the associated Legendre functions:

$$P_l(x)=\sum_{k=0}^{[l/2]}\frac{(2l-2k)!}{2^l(l-k)!k!(l-2k)!}(-1)^k x^{l-2k} \tag{21}$$

Since the highest power of x in $P_l(x)$ is x^l, we see that if $m>l$, $d^mP_l/dx^m=0$, so we must have $m\le l$. Further, since the m^{th} derivative of x^{l-2k} is $x^{l-2k-m}(l-2k)!/(l-2k-m)!$, the highest non-zero value of k in the sum is $[(l-m)/2]$ and we have (up to a normalization factor)

$$P_l^m(x) = (1-x^2)^{m/2}\frac{d^mP_l(x)}{dx^m} \tag{22}$$

$$= (1-x^2)^{m/2}\sum_{k=0}^{[(l-m)/2]}\frac{(2l-2k)!}{2^l(l-k)!k!(l-2k-m)!}(-1)^k x^{l-m-2k} \tag{23}$$

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