BESSEL FUNCTION SERIES

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The function

$$f(z)=\exp\left[\frac{\lambda}{2}\left(z-\frac{1}{z}\right)\right] \tag{1}$$

can be expanded in a Laurent series involving Bessel functions. To do this, we use the theorem that introduced Laurent series which states that a function can be expanded in the series

$$f(z)=\sum_{k=-\infty}^{\infty}a_k(z-z_0)^k \tag{2}$$

where the coefficients are given by the contour integral

$$a_k=\frac{1}{2\pi i}\int_C\frac{f(\zeta)}{(\zeta-z_0)^{k+1}}d\zeta \tag{3}$$

Here, we're interested in 1 in the domain $|z|>0$. As the function doesn't have any obvious expansion as a geometric series, we try to use 3 to work out the coefficients a_k. We'll use the contour $|z|=1$. The parametrization of this circle is given by

$$\zeta=e^{i\theta} \tag{4}$$

for $0\le\theta<2\pi$, and $z_0=0$. The contour integral is

$$\frac{1}{2\pi i}\int_C\frac{f(\zeta)}{\zeta^{k+1}}d\zeta=\frac{1}{2\pi i}\int_0^{2\pi}\frac{f(\theta)}{e^{(k+1)i\theta}}\frac{d\zeta}{d\theta}d\theta \tag{5}$$

From 1 we have

$$f(z)=\exp\left[\frac{\lambda}{2}\left(e^{i\theta}-e^{-i\theta}\right)\right] \tag{6}$$

$$=\exp(i\lambda\sin(\theta)) \tag{7}$$

Also

$$\frac{d\zeta}{d\theta} = ie^{i\theta} \tag{8}$$

so we have

$$\frac{1}{2\pi i}\int_0^{2\pi}\frac{f(\theta)}{e^{(k+1)i\theta}}\frac{d\zeta}{d\theta}d\theta = \frac{1}{2\pi i}\int_0^{2\pi}\exp\left(i\lambda\sin(\theta)\right)e^{-(k+1)i\theta}ie^{i\theta}d\theta \tag{9}$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\exp\left[-i\left(k\theta-\lambda\sin\theta\right)\right]d\theta \tag{10}$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\left[\cos\left(k\theta-\lambda\sin\theta\right)-i\sin\left(k\theta-\lambda\sin\theta\right)\right]d\theta \tag{11}$$

The second term actually integrates to zero[1], as we can see from the following argument. First, we note that $\sin(k\theta-\lambda\sin\theta)$ is an odd function relative to the point $\theta=\pi$. We can show this as follows.

$$\sin\left[k(\pi+\theta)-\lambda\sin(\pi+\theta)\right] = \sin\left[k\pi+k\theta-\lambda(\cos\pi\sin\theta)\right] \tag{12}$$

$$= \sin\left[k\pi+(k\theta+\lambda\sin\theta)\right] \tag{13}$$

$$= \cos(k\pi)\sin(k\theta+\lambda\sin\theta) \tag{14}$$

$$= (-1)^k\sin(k\theta+\lambda\sin\theta) \tag{15}$$

Now consider

$$\sin\left[k(\pi-\theta)-\lambda\sin(\pi-\theta)\right] = \sin\left[k\pi-k\theta-\lambda(-\cos\pi\sin\theta)\right] \tag{16}$$

$$= \sin\left[k\pi-(k\theta+\lambda\sin\theta)\right] \tag{17}$$

$$= \cos(k\pi)\left[-\sin(k\theta+\lambda\sin\theta)\right] \tag{18}$$

$$= -(-1)^k\sin(k\theta+\lambda\sin\theta) \tag{19}$$

From 15 and 19 we see that

$$\sin\left[k(\pi+\theta)-\lambda\sin(\pi+\theta)\right] = -\sin\left[k(\pi-\theta)-\lambda\sin(\pi-\theta)\right] \tag{20}$$

Thus any integral over an interval, such as $[0,2\pi]$, that is symmetric about $\theta=\pi$ is zero. We therefore have for the coefficients

$$a_k = \frac{1}{2\pi}\int_0^{2\pi}\cos(k\theta-\lambda\sin\theta)\,d\theta \tag{21}$$

This integral is the Bessel function of the first kind of order k.

[1] I realized this by looking at a plot of the function in Maple.