BINOMIAL THEOREM FOR COMPLEX NUMBERS

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Post date: 1 November 2024.

We present a proof of the binomial theorem for the expansion of a power of a binomial expression. There's nothing special about the fact that we're using complex numbers, but that's the most general form.

The theorem is stated as

$$(a+b)^n=\sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^k \tag{1}$$

where a and b are any complex numbers, and the *binomial coefficient* is defined as

$$\binom{n}{k}\equiv\frac{n!}{k!\,(n-k)!} \tag{2}$$

The easiest proof is by induction. That is, we show that the theorem is true for a particular value, say $n=1$. We then assume it's true for a general value of n and show that the result for $n+1$ follows from this.

For $n=1$, 1 says

$$(a+b)^1=\binom{1}{0}a^1b^0+\binom{1}{1}a^0b^1 \tag{3}$$

$$=a+b \tag{4}$$

which is certainly true. We now assume that 1 is true for a particular value of n, and then show that the formula is also valid for $n+1$. Expanding 1 we have

$$(a+b)^n=a^n+\binom{n}{1}a^{n-1}b+\ldots+\binom{n}{k}a^{n-k}b^k+\ldots+\binom{n}{n-1}ab^{n-1}+b^n \tag{5}$$

Multiplying both sides by $(a+b)$ we get

$$(a+b)^{n+1} = \binom{n}{0}a^{n+1} + \binom{n}{0}a^n b + \binom{n}{1}a^n b + \binom{n}{1}a^{n-1}b^2 + \binom{n}{2}a^{n-1}b^2 + \ldots + \tag{6}$$

$$\binom{n}{k}a^{n-k}b^{k+1} + \binom{n}{k+1}a^{n-k}b^{k+1} + \ldots + \tag{7}$$

$$\binom{n}{n-1}a^{n-n+1}b^n + \binom{n}{n}a^{n-n+1}b^n + \binom{n}{n}b^{n+1} \tag{8}$$

We note that the terms (apart the first and last terms) occur in pairs, with the general form of a pair as given in the middle line:

$$\binom{n}{k}a^{n-k}b^{k+1} + \binom{n}{k+1}a^{n-k}b^{k+1} \tag{9}$$

Here, the exponents of a are the same in the two terms, as are the exponents of b. We therefore need to work out the sum of binomial coefficients:

$$\binom{n}{k} + \binom{n}{k+1} = \frac{n!}{k!\,(n-k)!} + \frac{n!}{(k+1)!\,(n-k-1)!} \tag{10}$$

$$= \frac{n!\,[(k+1)+(n-k)]}{(k+1)!\,(n-k)!} \tag{11}$$

$$= \frac{(n+1)!}{(k+1)!\,(n-k)!} \tag{12}$$

$$= \binom{n+1}{k+1} \tag{13}$$

Thus we have

$$\binom{n}{k}a^{n-k}b^{k+1} + \binom{n}{k+1}a^{n-k}b^{k+1} = \binom{n+1}{k+1}a^{n-k}b^{k+1} \tag{14}$$

Adding in the first term from 6 and using the fact that $\binom{n}{0} = \binom{n+1}{0} = 1$, we have the term $\binom{n}{0}a^{n+1} = \binom{n+1}{0}a^{n+1}$. Similarly, the last term in 8 can be written as $\binom{n+1}{n+1}b^{n+1}$. Thus we can rewrite the general term in 14 as

$$\binom{n+1}{k}a^{n+1-k}b^k \tag{15}$$

where k runs from 0 to $n+1$. The overall formula becomes

$$(a+b)^{n+1} = \sum_{k=0}^{n+1}\binom{n+1}{k}a^{n+1-k}b^k \tag{16}$$

which is the binomial expansion for an exponent of $n+1$. QED.

Example 1. We can use 1 to expand the binomial $(2-i)^5$. We have

$$(2-i)^5 = \sum_{k=0}^{5}\binom{5}{k}2^{5-k}i^k \tag{17}$$
$$= 2^5 - 5\times 2^4 i + 10\times 2^3 i^2 - 10\times 2^2 i^3 + 5\times 2i^4 - i^5 \tag{18}$$
$$= 32 - 80i - 80 + 40i + 10 - i \tag{19}$$
$$= -38 - 41i \tag{20}$$

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