BOUNDARY VALUE PROBLEMS - CIRCLES

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One application of the complex logarithm is to find a harmonic function $\phi(z)$ (that is, a function that satisfies Laplace's equation $\phi_{xx}+\phi_{yy}=0$) subject to certain boundary conditions. Since the logarithm

$$\text{Log } z = \text{Log}\,|z| + i\text{Arg } z \tag{1}$$

is an analytic function over the complex plane, with the exception of its branch cut, we can use its real or imaginary part as a solution.

Suppose we have two concentric circles of radii r_1 and r_2 with $r_2 > r_1$, centred at the origin, and we wish to find a harmonic function $\phi(z)$ for the region between the circles, subject to boundary conditions imposed on the circles themselves. Further suppose that we require

$$\begin{aligned} \phi(r_1) &= B \\ \phi(r_2) &= A \end{aligned} \tag{2}$$

where A and B are constants. The geometry of the system is such that it has circular symmetry (that is, it's independent of Arg z), so we can try a general solution of the form

$$\phi(z) = a\text{Log}\,|z| + b \tag{3}$$

where a and b are constants to be determined by the boundary conditions.

From 2 we have

$$a\text{Log}r_1 + b = B \tag{4}$$

$$a\text{Log}r_2 + b = A \tag{5}$$

Subtracting the second equation from the first, we get

$$a(\text{Log}r_1 - \text{Log}r_2) = B - A \tag{6}$$

or

$$a = \frac{B-A}{\text{Log}r_1 - \text{Log}r_2} \tag{7}$$

Substituting back into 4 gives us

$$b = B - \frac{(B-A)\,\mathrm{Log}r_1}{\mathrm{Log}r_1 - \mathrm{Log}r_2} \tag{8}$$

$$= \frac{A\mathrm{Log}r_1 - B\mathrm{Log}r_2}{\mathrm{Log}r_1 - \mathrm{Log}r_2} \tag{9}$$

Therefore

$$\phi(z) = \frac{B-A}{\mathrm{Log}r_1 - \mathrm{Log}r_2}\mathrm{Log}\,|z| + \frac{A\mathrm{Log}r_1 - B\mathrm{Log}r_2}{\mathrm{Log}r_1 - \mathrm{Log}r_2} \tag{10}$$

Note that, since we don't use the argument of z here, we don't need to worry about branch cuts.

Example 1. Suppose we are given $r_1 = 1$, $r_2 = 2$, $B = 20$ and $A = 30$. Then

$$\phi(z) = \frac{-10}{0-\mathrm{Log}2}\mathrm{Log}\,|z| + \frac{30\times 0 - 20\mathrm{Log}2}{0-\mathrm{Log}2} \tag{11}$$

$$= \frac{10}{\mathrm{Log}2}\mathrm{Log}\,|z| + 20 \tag{12}$$

Example 2. Suppose we want to find a harmonic function that covers the interior of a disk of radius 2 and has the value 30 on the boundary. Since there is no inner circle, we'd expect that $\phi(z) = 30$ (a constant over the disk) to be a solution. We can verify this by setting $A = 30$ in 10 and taking the limit as $r_1 \to 0$. In this case, $\mathrm{Log}r_1 \to -\infty$, so the limit is

$$\phi(z) = \lim_{r_1\to 0}\frac{B-30}{\mathrm{Log}r_1 - \mathrm{Log}2}\mathrm{Log}\,|z| + \frac{30\mathrm{Log}r_1 - B\mathrm{Log}2}{\mathrm{Log}r_1 - \mathrm{Log}2} \tag{13}$$

$$= 0 + 30 \tag{14}$$

$$= 30 \tag{15}$$

Thus the formula 10 gives a sensible result in this limit.

Example 3. Now suppose we have two concentric circles of radii $r_1 = 1$ and $r_2 = 2$, but the common centre is now at the point $z_0 = 1 + i$. In this case, we can shift the circles' centre to the origin and consider

$$\phi(z) = a\mathrm{Log}\,(z - z_0) + b \tag{16}$$

The form of $\phi(z)$ is then the same as 10 with z replaced by $z - z_0 = z - 1 - i$ so we have

$$\phi(z) = \frac{B-A}{\text{Log}1-\text{Log}2}\text{Log}|z-1-i| + \frac{A\text{Log}1-B\text{Log}2}{\text{Log}1-\text{Log}2} \tag{17}$$

$$= \frac{A-B}{\text{Log}2}\text{Log}|z-1-i| + B \tag{18}$$

With $A=10$ and $B=0$, for example, we have

$$\phi(z) = \frac{10}{\text{Log}2}\text{Log}|z-1-i| \tag{19}$$

The value at $z=0$ is

$$\phi(0) = \frac{10}{\text{Log}2}\text{Log}|-1-i| \tag{20}$$

$$= \frac{10}{\text{Log}2}\text{Log}\sqrt{2} \tag{21}$$

$$= \frac{10}{\text{Log}2}\frac{\text{Log}2}{2} \tag{22}$$

$$= 5 \tag{23}$$

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