BOUNDARY VALUE PROBLEMS - WEDGES

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Boundary value problems involving circles can often be solved using the harmonic function $a\text{Log }|z|+b$, where a and b are constants. This is because a circle is a curve with constant radius, so the argument of the logarithm doesn't need to be used.

Conversely, we can solve boundary value problems involving wedges by using only the argument portion of the complex logarithm.

Example 1. Consider the setup in Fig. 1. The wedge of interest is the shaded section (which we assume to extend to infinity off to the right). The boundary at $\theta=-\frac{\pi}{4}$ has a constant value of $\phi=10$, while the boundary at $\theta=\frac{\pi}{4}$ has a constant value of $\phi=50$. As the region doesn't include the branch cut (the negative real axis), we can use the principal value Arg z.

We propose a solution of the form

$$\phi(z)=a\text{Arg }z+b \tag{1}$$

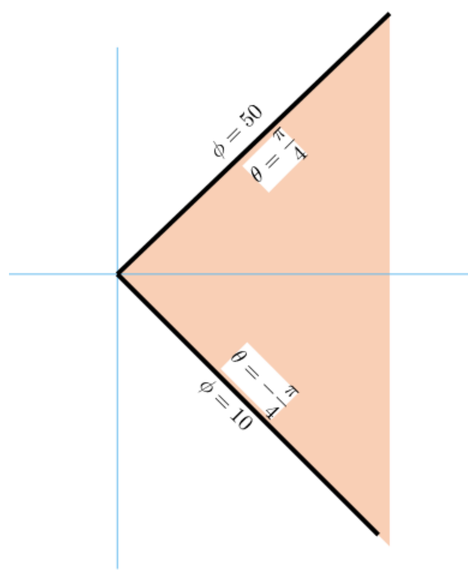

FIGURE 1. A wedge in the right half-plane.

where a and b are to be determined by the boundary conditions.

At $\theta = -\frac{\pi}{4}$, we have

$$\phi = -a\frac{\pi}{4} + b = 10 \tag{2}$$

while at $\theta = \frac{\pi}{4}$, we have

$$\phi = a\frac{\pi}{4} + b = 50 \tag{3}$$

Subtracting 2 from 3 we have

$$2a\frac{\pi}{4} = 40 \tag{4}$$

so

$$a = \frac{80}{\pi} \tag{5}$$

Substituting back, we find

$$b = 30 \tag{6}$$

The desired function is then

$$\phi(z) = \frac{80}{\pi}\text{Arg } z + 30 \tag{7}$$

Example 2. Consider the region shown in Fig. 2. Again, the region of interest is the shaded area, which has a corner at $z = 1 + i$, and boundary conditions along the two edges as shown. The shaded area is assumed to extend to infinity beyond the borders shown.

To solve this, we can first shift the system so the corner is at the origin by considering $\arg(z - 1 - i)$. However, this time the area contains the branch cut on the negative real axis, so we must use a different branch. We can do this by considering the branch which starts at $\theta = 0$ and wraps around to $\theta = 2\pi$, so the branch cut is now on the positive real axis. We refer to this branch as $\arg_0$. So the function we're looking for has the form

$$\phi(z) = a\arg_0(z - 1 - i) + b \tag{8}$$

The horizontal edge with $\phi = 10$ has $\arg_0 = 0$ and the vertical edge with $\phi = 0$ has $\arg_0 = \frac{3\pi}{2}$. The boundary conditions give us

$$a \times 0 + b = 10 \tag{9}$$

$$a \times \frac{3\pi}{2} + b = 0 \tag{10}$$

Therefore

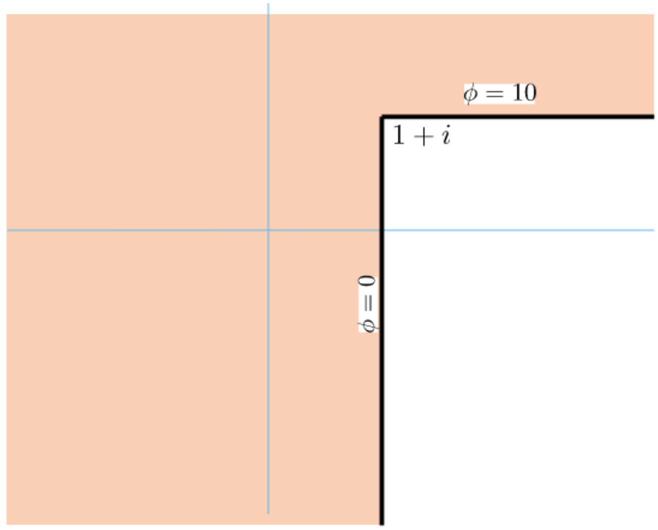

FIGURE 2. Corner wedge.

$$\begin{aligned} b &= 10 \\ a &= -\frac{20}{3\pi} \end{aligned} \tag{11}$$

The function is

$$\phi(z) = -\frac{20}{3\pi}\arg_0(z - 1 - i) + 10 \tag{12}$$

The value at $z = 0$ is

$$\phi(0) = -\frac{20}{3\pi}\arg_0(-1 - i) + 10 \tag{13}$$

$$= -\frac{20}{3\pi}\frac{5\pi}{4} + 10 \tag{14}$$

$$= -\frac{25}{3} + 10 \tag{15}$$

$$= \frac{5}{3} \tag{16}$$

Example 3. Consider the region to be the upper half plane, so the boundary is now the real axis. The boundary conditions are

$$\phi = \begin{cases} 0 & x < -1 \\ \pi & -1 < x < 2 \\ 0 & x > 2 \end{cases} \tag{17}$$

We can treat this as two wedges, on each of which the angle between the two sides is π (180°). We can translate each wedge to the origin and add up the contributions from each. That is, we're looking for a function of form

$$\phi = a_1 \text{Arg } (z+1) + a_2 \text{Arg } (z-2) + b \tag{18}$$

We have 3 constants to find, but we also have 3 conditions. For $x < -1$, both $z+1$ and $z-2$ have the argument π, so

$$a_1\pi + a_2\pi + b = 0 \tag{19}$$

For $-1 < x < 2$, Arg $(z+1) = 0$ and Arg $(z-2) = \pi$, so

$$a_2\pi + b = \pi \tag{20}$$

Finally, for $x > 2$, both arguments are 0, so

$$b = 0 \tag{21}$$

Thus

$$\begin{aligned} b &= 0 \\ a_2 &= 1 \\ a_1 &= -1 \end{aligned} \tag{22}$$

and the function is

$$\phi(z) = -\text{Arg } (z+1) + \text{Arg } (z-2) \tag{23}$$

At the point $z = 2 + 3i$ we have

$$\phi(2+3i) = -\text{Arg } (3+3i) + \text{Arg } (3i) \tag{24}$$

$$= -\frac{\pi}{4} + \frac{\pi}{2} \tag{25}$$

$$= \frac{\pi}{4} \tag{26}$$