BRANCH POINTS AND CUTS IN THE COMPLEX PLANE

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We've looked at contour integration in the complex plane as a technique for evaluating integrals of complex functions and finding infinite integrals of real functions.

In some cases, the complex functions that are to be integrated are multi-valued. As a preliminary to the contour integration of such functions, we'll look at the concepts of *branch points* and *branch cuts* here.

The stereotypical function that is used to introduce branch cuts in most books is the complex logarithm function $\log z$ which is defined so that

$$e^{\log z} = z \tag{1}$$

If z is real and positive, this reduces to the familiar real logarithm function. (Here I'm using natural logs, so the real natural log function is usually written as ln. In complex analysis, the term log is usually used, so be careful not to confuse it with base 10 logs.) To generalize it to complex numbers, we write z in modulus-argument form

$$z = re^{i\theta} \tag{2}$$

and apply the usual rules for taking a log of products and exponentials:

$$\begin{aligned}
\log z &= \log r + i\theta && (3)\\
&= \log r + i\arg z && (4)
\end{aligned}$$

To see where problems arise, suppose we start with z on the positive real axis and increase θ. Everything is fine until θ approaches 2π. When θ passes 2π, the original complex number z returns to its starting value, as given by 2. However, the imaginary component of $\log z$ continues to increase. This means that for the *same* complex number z, we can write down *two* different logarithms. In fact, every time we add another 2π to θ by going round the origin one full cycle, we generate yet *another* value for $\log z$, even though z itself doesn't change. Thus we have an infinite number of logarithms for any given complex number z (except for $z = 0$, which is undefined). That is

$$\log z = \log r + (\theta + 2\pi n)\, i \tag{5}$$

for any integer (positive, negative or zero) value n.

The problem is usually solved by restricting θ to one specific range covering 2π. We might, for example, require that $0 \le \theta < 2\pi$, although we could equally well require $-\pi < \theta \le \pi$ or in fact $\theta_0 \le \theta < \theta_0 + 2\pi$ for any value of θ_0.

For the log, this multivalued-ness occurs only if we follow a path that cycles around the origin. Any path that doesn't entirely circle the origin automatically restricts θ to an interval less than 2π in size. For example, suppose we pick a starting value of z in the first quadrant with $\theta = \frac{\pi}{4}$. If we now move z along a path that stays entirely within the first quadrant (the path can have any shape, even crossing itself, as long as it stays within the first quadrant), then $0 < \theta < \frac{\pi}{2}$ for the entire path so the log for any point on this path always has a unique value.

For $\log z$, the origin thus has a special property: moving z along any path that entirely circles the origin causes $\log z$ to become multivalued. A point having this property is called a *branch point*. We need to make this definition a bit more precise (mathematicians, of course, would do this with lots of symbols, but for physicists, the concept is what matters). The multivalued effect must occur provided we get close enough to the branch point. For example, suppose we picked a point X somewhere in the complex plane (not the origin). Then if we chose a closed path around X that was large enough to contain the origin as well, we would get a multivalued log. However, if we shrink the size of the path, keeping X inside the path, eventually we would exclude the origin and when that happens, the multivalued effect disappears and the log reverts to being single-valued. Thus X is not a branch point; only the origin is.

How large a region in the complex plane can we have while still retaining a single-valued log? The key is that whatever region we specify, it must not be possible to draw a closed path around the origin within that region. We can satisfy this requirement by drawing a curve with one end at the origin and the other at infinity, and then exclude all paths from crossing that curve. Usually the 'curve' is taken to be a straight line. For example, prohibiting the path from crossing the positive real axis solves the problem. Such an excluded curve is called a *branch cut*. There are usually an infinite number of possible branch cuts for any function that has one or more branch points. For the log, we could also choose the *negative* real axis, or the positive imaginary axis, or indeed *any* non-closed curve that starts at the origin and extends to infinity.

For some functions, infinity itself can be considered a branch point, although this can be difficult to understand at first. The idea is to think of a curve 'surrounding infinity', which can be done by thinking of an extremely large loop that encloses essentially the entire complex plane. If following such a loop causes the function to become multivalued, then infinity is a branch point. How large is 'extremely large'? Basically, any curve that encloses all points where the function does something odd is large enough.

For $\log z$, infinity is a branch point, since the only place where $\log z$ has any odd behaviour is at $z = 0$ itself, so our 'extremely large' loop just has to enclose the origin. With infinity included as a branch point, we can see that a branch cut is any path that connects the two branch points.

It's easier to understand branch points and cuts from a few examples.

Example 1. $f(z) = \log(z-1)$. Here, we can write the argument of the log as

$$z - 1 = \rho e^{i\phi} \tag{6}$$

$$\log(z-1) = \log\rho + i\phi \tag{7}$$

$$z = 1 + \rho e^{i\phi} \tag{8}$$

In this case, ϕ is the argument of z measured relative to the new 'origin' $1 + 0i$, and any closed path that circles this new origin will cause ϕ to increase in multiples of 2π, leaving $z - 1$ unchanged but increasing the value of the log. Thus $z = 1$ is a branch point and, since any closed path around this branch point causes a multivalued $f(z)$, $z = \infty$ is also a branch point. A suitable branch cut is a curve with one end at $z = 1$ and extending to infinity, so, for example, the real axis with $x \geq 1$ would serve as a branch cut, as would $x \leq 1$.

Example 2. $f(z) = \log\frac{z+1}{z-1}$. Now we have

$$\log\frac{z+1}{z-1} = \log(z+1) - \log(z-1) \tag{9}$$

By the same reasoning as in Example 1, $z = -1$ and $z = +1$ are both branch points. Suppose we draw a closed curve around $z = -1$ but ensure that this curve is sufficiently close to $z = -1$ that it excludes $z = +1$. Then the argument of $\log(z+1)$ will increase by 2π while the argument of $\log(z-1)$ will not change, so the overall log will increase by $2\pi i$. Doing the same thing for $z = +1$ will increase the argument of $\log(z-1)$ by 2π so the overall log will *decrease* by $2\pi i$ (because of the minus sign in front of $\log(z-1)$), so again the overall log is multivalued.

How about infinity? To test this, we draw a closed curve big enough to encompass *both* $z=\pm1$. This causes the argument of both terms to increase by 2π but because we're taking the difference, these two terms cancel out and the overall log doesn't change. Thus in this case, infinity is *not* a branch point.

A suitable branch cut in this case is any curve which prevents a path from circling one branch point but excluding the other one. Thus any curve connecting $z=-1$ and $z=+1$ would do, so for example, the real axis segment $-1\le x\le+1$ would serve as a branch cut. We could also select the two segments $x\le-1$ and $x\ge+1$, although that would also exclude paths which encircle both branch points.

Example 3. $f(z)=\sqrt{z}$. We have

$$\sqrt{z} = \left[re^{i\theta}\right]^{1/2} \tag{10}$$

$$= r^{1/2}e^{i\theta/2} \tag{11}$$

Circling the origin once ($\theta\to\theta+2\pi$) gives the negative square root

$$\sqrt{z}=r^{1/2}e^{i\theta/2}e^{i\pi}=-r^{1/2}e^{i\theta/2} \tag{12}$$

Circling the origin twice ($\theta\to\theta+4\pi$) repeats the positive root

$$\sqrt{z}=r^{1/2}e^{i\theta/2}e^{i2\pi}=r^{1/2}e^{i\theta/2} \tag{13}$$

Circling the origin again just repeats the process, so that $\sqrt{z}$ alternates between the negative and positive roots. Thus, unlike the log which had an infinite number of possible values for any given value of z, the square root has only two possible values. There are two branch points at $z=0$ and $z=\infty$, and the same branch cuts that work with the log will work here too.

Branch cuts must also be taken into account when doing contour integration, as the contours must be careful not to cross the branch cuts.

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