BRANCHES OF LOGARITHMS

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog and include the title or URL of this post in your comment.

Post date: 29 December 2024.

The principal value $\text{Log}z$ of the logarithm of a complex number is defined by

$$\text{Log } z = \text{Log}\,|z| + i\text{Arg}z \tag{1}$$

where Arg z is the principal argument, lying in the interval $(-\pi,\pi]$. The more general form (with a small ℓ) is defined by

$$\log z = \text{Log}\,|z| + i\arg z \tag{2}$$

where

$$\arg z = \text{Arg}z + 2k\pi \tag{3}$$

where k is any integer.

The form 1 is the *principal branch* of the logarithm. Other branches of the logarithm can be found by adding a multiple of 2π to the argument.

The principal branch of a logarithm has a *branch point* at $z = 0$, since the logarithm is not defined there. It has a *branch cut* along the negative real axis, since if we cross this line, the principal argument jumps from π to $-\pi$ (going counterclockwise). The same branch point and branch cut exist for any branch, since the same discontinuity in the argument occurs in all branches.

The principal logarithm is analytic over the entire complex plane except for its branch cut. Logarithms of functions of z also have domains of analyticity, which can be found by determining their branch points and cuts.

Example 1. Find the domain of analyticity for

$$f(z) = \text{Log}\,(4 + i - z) \tag{4}$$

The branch point is at

$$4 + i - z = 0 \tag{5}$$

or

$$z = 4 + i \tag{6}$$

The branch cut occurs for all values of the function's argument that give non-positive real numbers. Writing $z = x + iy$ we have

$$4 + i - z = 4 - x + i(1 - y) \tag{7}$$

This gives non-positive real values if

$$\begin{aligned} y &= 1 \\ x &\geq 4 \end{aligned} \tag{8}$$

The branch cut is therefore a horizontal line at $y = 1$ and covering the interval $[4, \infty)$.

The derivative is

$$f'(z) = \frac{-1}{4 + i - z} \tag{9}$$

We can choose a different location for a branch cut.

Example 2. Consider

$$f(z) = \log(2z - 1) \tag{10}$$

For the standard branch cut, we find a branch point at

$$2z - 1 = 0 \tag{11}$$

or

$$z = \frac{1}{2} \tag{12}$$

We get non-positive real values if

$$2x - 1 + 2iy \leq 0 \tag{13}$$

so

$$\begin{aligned} y &= 0 \\ x &\leq \frac{1}{2} \end{aligned} \tag{14}$$

Thus the standard branch cut is the real axis over the interval $x \in \left(-\infty, \frac{1}{2}\right]$.

If we want the branch cut to be over the interval $x \in \left[\frac{1}{2}, \infty\right)$ we can add π to the argument. That is

$$\log(2z-1) = \text{Log}|2z-1| + i(\text{Arg}(2z-1)+\pi) \tag{15}$$

The argument now spans the interval $(0, 2\pi]$ so the argument is discontinuous if we cross the new branch cut.

We can also make the branch cut a vertical line at $x = \frac{1}{2}$ and $y \geq 0$, by adding $\frac{3\pi}{2}$ to the principal argument. That is

$$\log(2z-1) = \text{Log}|2z-1| + i\left(\text{Arg}(2z-1) + \frac{3\pi}{2}\right) \tag{16}$$

The argument now spans the interval $\left(\frac{3\pi}{2}, \frac{7\pi}{2}\right]$.

Note that however we do it, we must always include the point $z = \frac{1}{2}$ as the branch point, since that is where $2z-1=0$ and the logarithm is undefined.

Example 3. Branch points and cuts of

$$f(z) = \log\left(z^2+2z+3\right) \tag{17}$$

The branch points are at the zeroes of z^2+2z+3, which are

$$z_0 = -1 \pm \sqrt{2}i \tag{18}$$

The branch cuts are intervals where z^2+2z+3 is real and negative. Substituting $z = x+iy$ we get

$$z^2+2z+3 = \left[x^2-y^2+2x+3\right] + (2xy+2y)\,i \tag{19}$$

To get real values, we must have either $x=-1$ or $y=0$. If $y=0$, then we require $x^2+2x+3<0$. However this is the same polynomial we started with, except that x is real rather than complex. However, we know from 18 that the roots of this polynomial are complex, so its graph never crosses the x axis, which means that it is positive for all x. Therefore, we must have $x=-1$ in 19, giving us

$$2-y^2 \leq 0 \tag{20}$$

so

$$\begin{aligned} y &\geq \sqrt{2} \\ y &\leq -\sqrt{2} \end{aligned} \tag{21}$$

Thus the branch cuts are the two vertical lines at $x=-1$ that cover the y intervals $\left[\sqrt{2}, \infty\right)$ and $\left(-\infty, -\sqrt{2}\right]$. Crossing either of these branch cuts causes the argument to jump by 2π.

PINGBACKS

Pingback: Boundary value problems - circles
Pingback: Complex powers
Pingback: Cauchy's integral theorem