CAUCHY ESTIMATES - EXAMPLES

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Post date: 31 January 2025.

The Cauchy estimates theorem states

Theorem 1. *(Cauchy estimates) Suppose that $f(z)$ is analytic inside and on a circle C_R of radius R, centred at z_0. If $|f(z)| \le M$ for all z on C_R, then the derivatives of f at z_0 satisfy*

$$\left|f^{(n)}(z_0)\right| \le \frac{n!M}{R^n} \tag{1}$$

Here are a couple of examples using this theorem.

Example 1. Suppose that a function $f(z)$ is analytic and bounded by M in the disk $|z| \le r$. What can we say about the derivatives at a given point z_0 inside this disk? To apply 1 we draw a circle centred at z_0 such that this circle is tangent to the circle $|z| = r$. If we draw a line from the origin through z_0, then the distance along this line from z_0 to the circle $|z| = r$ is $r - |z_0|$. Thus we can apply 1 with $R = r - |z_0|$ and we get

$$\left|f^{(n)}(z_0)\right| \le \frac{n!M}{(r - |z_0|)^n} \tag{2}$$

Example 2. Let

$$p(z) = a_0 + a_1 z + \ldots + a_n z^n \tag{3}$$

be a polynomial. Suppose that

$$\max_{|z|=1} |p(z)| = M \tag{4}$$

That is, the maximum modulus of $p(z)$ is M on the circle $|z| = 1$. Consider $z_0 = 0$. Then

$$\begin{aligned} p(0) &= a_0 \\ p'(0) &= a_1 \\ p^{(2)}(0) &= 2a_2 \\ &\vdots \\ p^{(n)}(0) &= n!a_n \end{aligned} \tag{5}$$

Then from 1 with $R = 1$ we have

$$\left|p^{(k)}(0)\right| = k!\,|a_k| \leq k!M \tag{6}$$

Therefore all the coefficients satisfy the bound

$$|a_k| \leq M \tag{7}$$