CAUCHY ESTIMATES

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The Cauchy estimates apply to derivatives of an analytic function $f(z)$. They can be derived from the Cauchy integral formula and the formula for the upper bound on a contour integral. The theorem is

Theorem 1. *(Cauchy estimates) Suppose that $f(z)$ is analytic inside and on a circle C_R of radius R, centred at z_0. If $|f(z)| \le M$ for all z on C_R, then the derivatives of f at z_0 satisfy*

$$\left|f^{(n)}(z_0)\right| \le \frac{n!M}{R^n} \tag{1}$$

The proof is given in Saff and Snider's book, section 4.6.

Example 1. Let

$$f(z) = \frac{1}{(1-z)^2} \tag{2}$$

and let $0 < R < 1$. Consider a circle C_R of radius R centred at $z_0 = 0$. Then on C_R the maximum of $|f|$ is when $|1-z|$ has its minimum. Since $|1-z| = |z-1|$, this represents a circle centred at $z = 1$. We therefore want the shortest distance of a circle centred at $z = 1$ to the circle $|z| = R$. This occurs on the real axis for a circle centred at $z = 1$ of radius $1-R$. To see this, remember that $0 < R < 1$, so the circle $|z| = R$ extends out to $z = R$ along the positive real axis. Thus the radius of a circle $|z-1|$ will just touch the circle $|z| = R$ if the radius of the $|z-1|$ circle is $1-R$.

Thus

$$\max_{|z|=R} |f(z)| = \frac{1}{(1-R)^2} \tag{3}$$

Consider the derivatives of 2 at $z_0 = 0$. We have

$$f'(z) = \frac{2}{(1-z)^3} \tag{4}$$

$$f^{(2)}(z) = \frac{3\times 2}{(1-z)^4} \tag{5}$$

$$\vdots$$

$$f^{(n)}(z) = \frac{(n+1)!}{(1-z)^{n+1}} \tag{6}$$

At $z_0 = 0$, we have therefore

$$f^{(n)}(0) = (n+1)! \tag{7}$$

and from 1 we have

$$(n+1)! \le \frac{n!}{R^n}\frac{1}{(1-R)^2} \tag{8}$$

Remember that $0 < R < 1$, so the denominator will always be less than 1, ensuring that the RHS is bigger than $n!$, so this is a reasonable estimate.

Example 2. Suppose $f(z)$ is analytic in $|z| < 1$ and that

$$|f(z)| < \frac{1}{1-|z|} \tag{9}$$

Consider again the circle $|z| = R$ for $0 < R < 1$. On the circle, $z = Re^{i\theta}$, so $|z| = R$ and thus on the circle

$$\frac{1}{1-|z|} = \frac{1}{1-R} \tag{10}$$

Therefore

$$\max_{|z|=R} |f(z)| < \frac{1}{1-R} \tag{11}$$

From 1 we have

$$\left|f^{(n)}(0)\right| \le \frac{n!}{R^n(1-R)} \tag{12}$$

We can find the value of R for which this upper bound is smallest by the usual technique of taking the derivative and setting it to zero. We have

$$\frac{d}{dR}\frac{1}{R^n(1-R)} = -\frac{n}{R^{n+1}(1-R)} + \frac{1}{R^n(1-R)^2} \tag{13}$$

Setting to zero, we get

$$-\frac{n}{R^{n+1}(1-R)}+\frac{1}{R^{n}(1-R)^{2}}=0 \tag{14}$$

$$-\frac{n}{R}+\frac{1}{1-R}=0 \tag{15}$$

$$n(R-1)+R=0 \tag{16}$$

$$R=\frac{n}{n+1} \tag{17}$$

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