CAUCHY PRODUCT OF TAYLOR SERIES

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If we know the Taylor series for two analytic functions $f(z)$ and $g(z)$, we can find the Taylor series of the product fg by using the *Cauchy product*. We have, for the Taylor series about $z_0 = 0$:

$$f(z) = \sum_{j=0}^{\infty} a_j z^j \tag{1}$$

$$g(z) = \sum_{k=0}^{\infty} b_k z^k \tag{2}$$

The Cauchy product involves multiplying the two series term by term and collecting terms with the same power of z. This gives

$$fg = \sum_{\ell=0}^{\infty} c_\ell z^\ell = a_0 b_0 + (a_1 b_0 + a_0 b_1) z + (a_2 b_0 + a_1 b_1 + a_0 b_2) z^2 + \ldots \tag{3}$$

The general term for the coefficient c_j is

$$c_j = a_j b_0 + a_{j-1} b_1 + \ldots + a_1 b_{j-1} + a_0 b_j \tag{4}$$

$$= \sum_{\ell=0}^{j} a_{j-\ell} b_\ell \tag{5}$$

It is a theorem (proved in Saff and Snider's Section 5.2) that the Taylor series given by 3 is actually the correct series for the product of two functions.

Example 1. Find the series for $f(z) = e^z \cos z$. The two series to be multiplied are

$$e^{z}=\sum_{j=0}^{\infty}a_{j}z^{j}=\sum_{j=0}^{\infty}\frac{z^{j}}{j!} \tag{6}$$

$$\cos z=\sum_{k=0}^{\infty}b_{k}z^{k}=\sum_{k=0}^{\infty}(-1)^{k}\frac{z^{2k}}{(2k)!} \tag{7}$$

Therefore

$$\begin{aligned}
a_0 &= 1\\
a_1 &= 1\\
a_2 &= \frac{1}{2!}=\frac{1}{2}\\
a_3 &= \frac{1}{3!}=\frac{1}{6}\\
b_0 &= 1\\
b_1 &= 0\\
b_2 &= -\frac{1}{2}\\
b_3 &= 0
\end{aligned} \tag{8}$$

[Note that in calculating the b_k coefficients, the subscript k refers to the power of z in the series, so $b_1=0$ since there is no term in z in the series for $\cos z$.]

We can find the first 3 terms for $f(z)$ as

$$c_0 = a_0b_0 = 1 \tag{9}$$

$$c_1 = a_1b_0+a_0b_1 = 1 \tag{10}$$

$$c_2 = a_2b_0+a_1b_1+a_0b_2 = \frac{1}{2}+0-\frac{1}{2}=0 \tag{11}$$

$$c_3 = \sum_{\ell=0}^{3}a_{3-\ell}b_{\ell} = \frac{1}{6}+0-\frac{1}{2}+0=-\frac{1}{3} \tag{12}$$

The series starts with

$$e^{z}\cos z = 1+z-\frac{z^{3}}{3}+\ldots \tag{13}$$

Example 2. Series for $f(z)=\frac{e^{z}}{z-1}$. The two series are

$$e^z = \sum_{j=0}^{\infty} a_j z^j = \sum_{j=0}^{\infty} \frac{z^j}{j!} \tag{14}$$

$$\frac{1}{z-1} = \sum_{k=0}^{\infty} b_k z^k = -\sum_{k=0}^{\infty} z^k \tag{15}$$

Therefore

$$\begin{aligned} a_0 &= 1 \\ a_1 &= 1 \\ a_2 &= \frac{1}{2!} = \frac{1}{2} \\ a_3 &= \frac{1}{3!} = \frac{1}{6} \\ b_0 &= -1 \\ b_1 &= -1 \\ b_2 &= -1 \\ b_3 &= -1 \end{aligned} \tag{16}$$

The coefficients are

$$c_0 = a_0 b_0 = -1 \tag{17}$$

$$c_1 = a_1 b_0 + a_0 b_1 = -2 \tag{18}$$

$$c_2 = a_2 b_0 + a_1 b_1 + a_0 b_2 = -1 - 1 - \frac{1}{2} = -\frac{5}{2} \tag{19}$$

$$c_3 = \sum_{\ell=0}^{3} a_{3-\ell} b_\ell = -1 - 1 - \frac{1}{2} - \frac{1}{6} = -\frac{8}{3} \tag{20}$$

The series starts with

$$\frac{e^z}{z-1} = -1 - 2z - \frac{5}{2} z^2 - \frac{8}{3} z^3 + \dots \tag{21}$$

Example 3. Series for $\sec z$. Solving this one takes a slight trick. We write

$$\sec z = \sum_{j=0}^{\infty} a_j z^j \tag{22}$$

Next we observe that

$$\sec z = \frac{1}{\cos z} \tag{23}$$

and we know the series for $\cos z$:

$$\cos z=\sum_{k=0}^{\infty}b_{k}z^{k}=\sum_{k=0}^{\infty}(-1)^{k}\frac{z^{2k}}{(2k)!} \tag{24}$$

Therefore

$$\begin{aligned}(\sec z)(\cos z)=&\left(a_{0}+a_{1}z+a_{2}z^{2}+a_{3}z^{3}+a_{4}z^{4}\right)\times\\&\left(1+0-\frac{z^{2}}{2}+0+\frac{z^{4}}{24}+\ldots\right)\end{aligned} \tag{25}$$

Multiplying out we get

$$\begin{aligned}a_{0}\times1&=1\\\text{thus}\quad a_{0}&=1\\a_{1}\times1+a_{0}\times0&=0\\\text{thus}\quad a_{1}&=0\\a_{2}\times1+a_{1}\times0+a_{0}\times\left(-\frac{1}{2}\right)&=0\\\text{thus}\quad a_{2}&=\frac{a_{0}}{2}=\frac{1}{2}\\a_{3}\times1+a_{2}\times0+a_{1}\times\left(-\frac{1}{2}\right)+a_{0}\times0&=0\\\text{thus}\quad a_{3}&=0\\a_{4}\times1+a_{3}\times0+a_{2}\times\left(-\frac{1}{2}\right)+a_{1}\times0+a_{0}\times\frac{1}{24}&=0\\\text{thus}\quad a_{4}&=\frac{a_{2}}{2}-\frac{a_{0}}{24}=\frac{5}{24}\end{aligned} \tag{26}$$

Thus we have

$$\sec z=1+\frac{1}{2}z^{2}+\frac{5}{24}z^{4}+\ldots \tag{27}$$

Example 4. $f(z)=\tanh z$. We use the same method as in Example 3, noting that

$$\tanh z=\frac{\sinh z}{\cosh z} \tag{28}$$

The two series on the RHS are

$$\sinh z = z + \frac{z^3}{3!} + \frac{z^5}{5!} + \dots \tag{29}$$

$$\cosh z = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \dots \tag{30}$$

We have, working with the first few terms in each series:

$$(\tanh z)(\cosh z) = \left[a_0 + a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4 + a_5 z^5\right]\left[1 + \frac{z^2}{2!} + \frac{z^4}{4!}\right] \tag{31}$$

$$= \sinh z = z + \frac{z^3}{3!} + \frac{z^5}{5!} \tag{32}$$

Therefore

$$\begin{aligned} a_0 \times 1 &= 0 \\ \text{thus} \quad a_0 &= 0 \\ a_1 \times 1 + a_0 \times 0 &= 1 \\ \text{thus} \quad a_1 &= 1 \\ a_2 \times 1 + a_1 \times 0 + a_0 \times \frac{1}{2} &= 0 \\ \text{thus} \quad a_2 &= 0 \\ a_3 \times 1 + a_2 \times 0 + a_1 \times \frac{1}{2} + a_0 \times 0 &= \frac{1}{3!} \\ \text{thus} \quad a_3 &= -\frac{1}{3} \end{aligned} \tag{33}$$

We can continue for another couple of steps (I used Maple to do this) to find

$$\tanh z = z - \frac{1}{3}z^3 + \frac{2}{15}z^5 + \dots \tag{34}$$